FINITE ELEMENT : MATRIX FORMULATION

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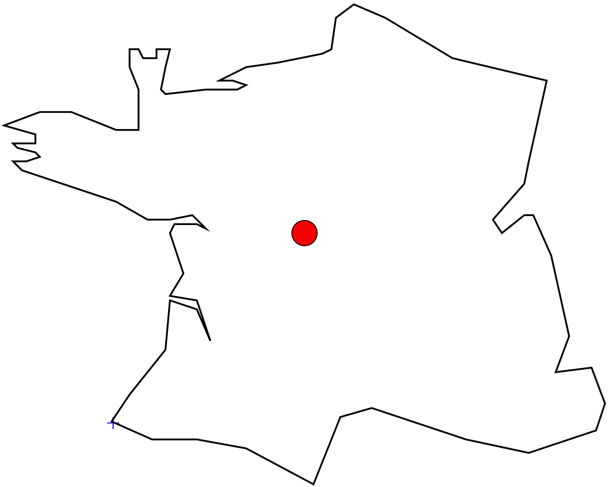
Contents

Discrete *versus* continuous

Element
- Interpolation
- Element list

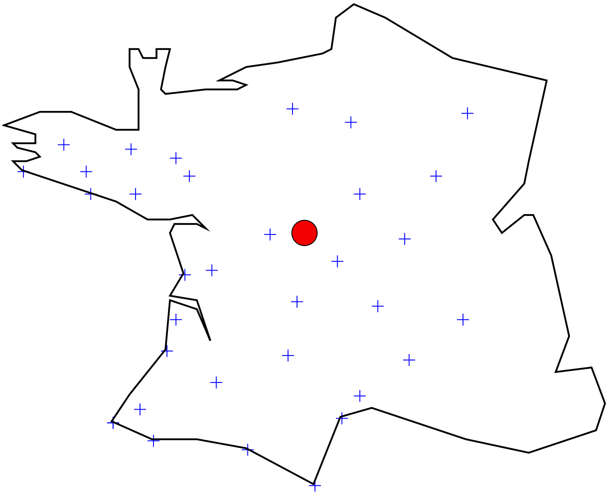
Global problem
- Formulation
- Matrix formulation
- Algorithm

Continuous$\longrightarrow$ Discrete$\longrightarrow$Continuous



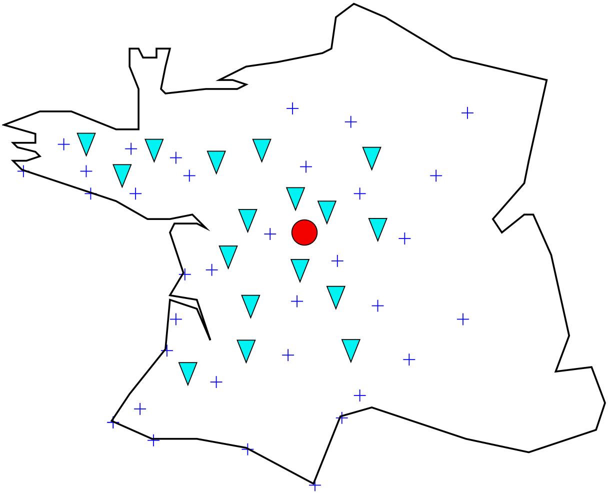
How much rain ?

Continuous→ Discrete→Continuous



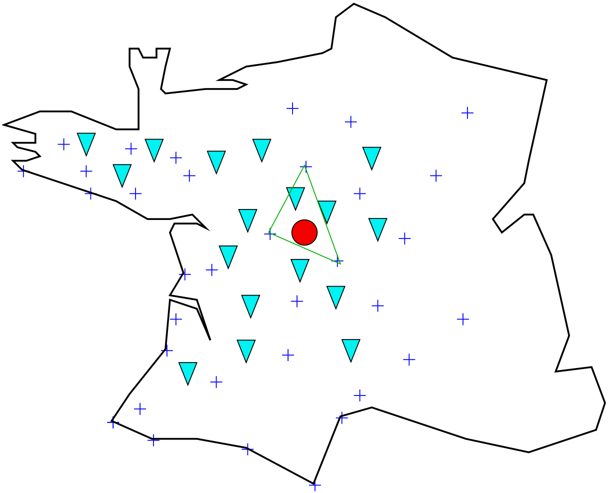
Geometry discretization

Continuous→ Discrete→Continuous



Unknown field discretization

Continuous→ Discrete→Continuous



Use elements

Finite Element Discretization

Replace continuum formulation by a discrete representation for unknowns and geometry

- Unknown field:

$$\underline{\mathbf{u}}^e(M) = \sum_i N_i^e(M)\underline{\mathbf{q}}_i^e$$

- Geometry:

$$\underline{\mathbf{x}}(M) = \sum_i N_i^{*e}(M)\underline{\mathbf{x}}(P_i)$$

Interpolation functions N_i^e and *shape* functions N_i^{*e} such as:

$$\forall M, \quad \sum_i N_i^e(M) = 1 \text{ and } N_i^e(P_j) = \delta_{ij}$$

Isoparametric elements iff $N_i^e \equiv N_i^{*e}$

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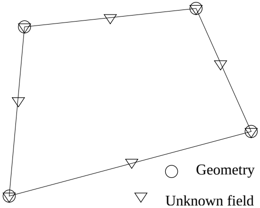
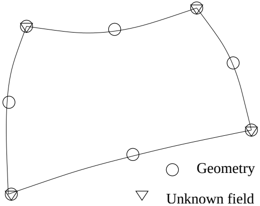
2D-mapping

Subparametric element



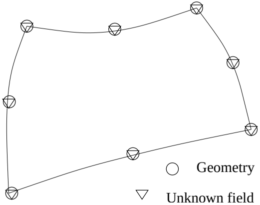
more field nodes than geometrical nodes

Superparametric element



more geometrical nodes than field nodes

Isoparametric element



same number of geom and field nodes

Rigid body displacement not represented for superparametric element that has nonlinear edges !

The location of the node at the middle of the edge is critical for quadratic edges

Shape function matrix, [N] – Deformation matrix, [B]

- Field $\underline{\mathbf{u}}$, T, C
- Gradient $\underset{\sim}{\varepsilon}$, $\underline{grad}(T)$,...
- Constitutive equations $\underset{\sim}{\sigma} = \underset{\approx}{\mathbf{\Lambda}} : \underset{\sim}{\varepsilon}$, $\underline{q} = -k\underline{grad}(T)$
- Conservation $\underline{div}(\underset{\sim}{\sigma}) + \underline{\mathbf{f}} = 0$, ...

First step: express the continuous field and its gradient wrt the discretized vector

Deformation matrix [B] (1)

- Knowing:

$$\underline{\mathbf{u}}^e(M) = \sum_i N_i^e(M)\underline{\mathbf{q}}_i^e$$

- Deformation can be obtained from the nodal *displacements*, for instance in 2D, small strain:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial N_1(M)}{\partial x}\, q_{1x}^e + \frac{\partial N_2(M)}{\partial x}\, q_{2x}^e + \dots$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial N_1(M)}{\partial y}\, q_{1y}^e + \frac{\partial N_2(M)}{\partial y}\, q_{2y}^e + \dots$$

$$2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{\partial N_1(M)}{\partial y}\, q_{1x}^e + \frac{\partial N_2(M)}{\partial x}\, q_{1y}^e + \dots$$

Deformation matrix [B] (2)

Matrix form, 4-node quadrilateral

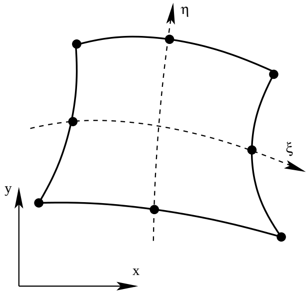
$$\{u\}^e = [N]^T \{q\}^e = \begin{pmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{pmatrix} \begin{pmatrix} q^e_{1x} \\ q^e_{1y} \\ \dots \\ q^e_{4y} \end{pmatrix}$$

$$\{\varepsilon\}^e = [B]^T \{q\}^e$$

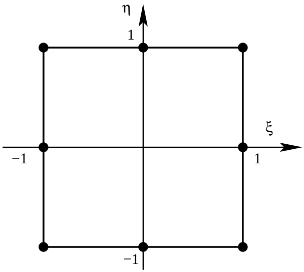
$$= \begin{pmatrix} N_{1,x} & 0 & N_{2,x} & 0 & N_{3,x} & 0 & N_{4,x} & 0 \\ 0 & N_{1,y} & 0 & N_{2,y} & 0 & N_{3,y} & 0 & N_{4,y} \\ N_{1,y} & N_{1,x} & N_{2,y} & N_{2,x} & N_{3,y} & N_{3,x} & N_{4,y} & N_{4,x} \end{pmatrix} \begin{pmatrix} q^e_{1x} \\ q^e_{1y} \\ \dots \\ q^e_{4y} \end{pmatrix}$$

Shear term taken as $\gamma = 2\varepsilon_{12}$

Reference element



Actual geometry
Physical space $(x,\ y)$



Reference element
Parent space $(\xi,\ \eta)$

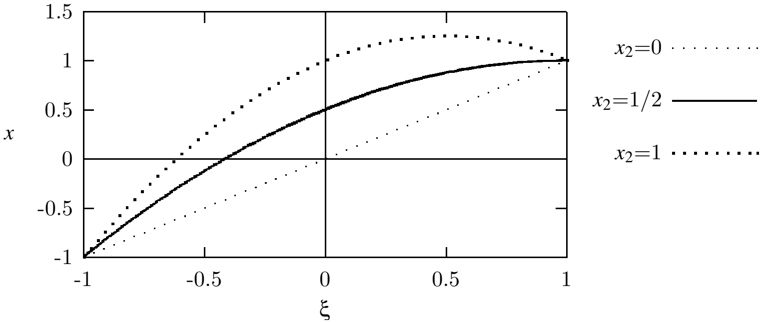
$$\int_{\Omega} f(x,y)dxdy = \int_{-1}^{+1}\int_{-1}^{+1} f_*(\xi,\eta)Jd\xi d\eta$$

J is the determinant of the partial derivatives $\partial x/\partial\xi\ldots$ matrix

Remarks on geometrical mapping

- The values on an edge depends only on the nodal values on the same edge (linear interpolation equal to zero on each side for 2-node lines, parabolic interpolation equal to zero for 3 points for 3-node lines)
- Continuity...
- The mid node is used to allow non linear geometries
- Limits in the admissible mapping for avoiding singularities

Mapping of a 3-node line



Physical segment:	x_1=-1	x_3=1	$-1 \leqslant x_2 \leqslant 1$
Parent segment:	ξ_1=-1	ξ_3=1	ξ_2=0

$$x = \xi + \left(1 - \xi^2\right) x_2$$

Jacobian and inverse jacobian matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = [J] \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}$$

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = [J]^{-1} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Since $(x,\ y)$ are known from $N_i(\xi, \eta)$ and x_i,
$[J]^{-1}$ is computed from the known quantities in $[J]$, using also:

$$J = Det\left([J]\right) = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}$$

Expression of the inverse jacobian matrix

$$[J]^{-1} = \frac{1}{J} \begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{pmatrix}$$

- For a rectangle $[\pm a, \pm b]$ in the "real world", the mapping function is the same for any point inside the rectangle. The jacobian is a diagonal matrix, with $\partial x/\partial \xi = a$, $\partial y/\partial \eta = b$, and the determinant value is ab
- For any other shape, the "mapping" changes according to the location in the element
- For computing $[B]$, one has to consider $\partial N_i/\partial x$ and $\partial N_i/\partial y$:

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad \text{then} \qquad \begin{pmatrix} \partial N_i/\partial x \\ \partial N_i/\partial y \end{pmatrix} = [J]^{-1} \begin{pmatrix} \partial N_i/\partial \xi \\ \partial N_i/\partial \eta \end{pmatrix}$$
$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

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2D solid elements

Type	shape	interpol of disp	# of nodes	polynom terms
C2D3	tri	lin	3	$1, \xi, \eta$
C2D4	quad	lin	4	$1, \xi, \eta, \xi\eta$
C2D6	tri	quad	6	$1, \xi, \eta, \xi^2, \xi\eta, \eta^2$
C2D8	quad	quad	8	$1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^2\eta, \xi\eta^2$
C2D9	quad	quad	9	$1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^2\eta, \xi\eta^2, \xi^2\eta^2$

3D solid elements

Type	shape	interpol of disp	# of nodes	polynom terms
C3D4	tetra	lin	4	$1, \xi, \eta, \zeta$
C3D6	tri prism	lin	6	$1, \xi, \eta, \zeta, \xi\eta, \eta\zeta$
C3D8	hexa	lin	8	$1, \xi, \eta, \zeta, \xi\eta, \eta\zeta, \zeta\xi, \xi\eta\zeta$
C3D10	tetra	quad	10	$1, \xi, \eta, \zeta, \xi^2, \xi\eta, \eta^2, \eta\zeta, \zeta^2, \zeta\xi$
C3D15	tri prism	quad	15	$1, \xi, \eta, \zeta, \xi\eta, \eta\zeta, \xi^2\zeta, \xi\eta\zeta, \eta^2\zeta, \zeta^2,$ $\xi\zeta^2, \eta\zeta^2, \xi^2\zeta^2, \xi\eta\zeta^2, \eta^2\zeta^2$
C3D20	hexa	quad	20	$1, \xi, \eta, \zeta, \xi^2, \xi\eta, \eta^2, \eta\zeta, \zeta^2, \zeta\xi,$ $\xi^2\eta, \xi\eta^2, \eta^2\zeta, \eta\zeta^2, \xi\zeta^2, \xi^2\zeta, \xi\eta\zeta,$ $\xi^2\eta\zeta, \xi\eta^2\zeta, \xi\eta\zeta^2$
C3D27	hexa	quad	27	$\xi^i\eta^j\zeta^k, (i, j, k) \in 0, 1, 2$

Isoparametric representation

Example: 2D plane stress elements with n *nodes*

- Element geometry

$$1 = \sum_{i=1}^{n} N_i \qquad x = \sum_{i=1}^{n} N_i x_i \qquad y = \sum_{i=1}^{n} N_i y_i$$

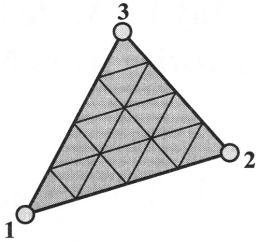
- Displacement interpolation

$$u_x = \sum_{i=1}^{n} N_i u_{xi} \qquad u_y = \sum_{i=1}^{n} N_i u_y i$$

- Matrix form

$$\begin{pmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \\ y_1 & y_2 & y_3 & \dots & y_n \\ u_{x1} & u_{x2} & u_{x3} & \dots & u_{xn} \\ u_{y1} & u_{y2} & u_{y3} & \dots & u_{yn} \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ . \\ . \\ N_n \end{pmatrix}$$

The linear triangle



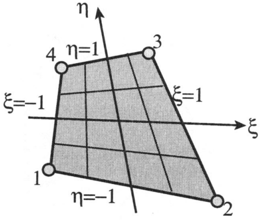
$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \end{bmatrix}$$

$$N_1^{(e)} = \zeta_1, \qquad N_2^{(e)} = \zeta_2, \qquad N_3^{(e)} = \zeta_3$$

Terms in 1, ξ, η

IFEM–Felippa

The bilinear quad



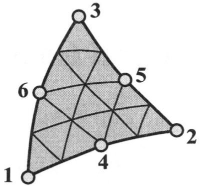
$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \end{bmatrix}$$

$$\begin{aligned} N_1^{(e)} &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ N_2^{(e)} &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ N_3^{(e)} &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N_4^{(e)} &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}$$

IFEM–Felippa

Terms in 1, ξ, η, $\xi\eta$

The quadratic triangle



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \\ N_5^{(e)} \\ N_6^{(e)} \end{bmatrix}$$

$$N_1^{(e)} = \zeta_1(2\zeta_1 - 1) \qquad N_4^{(e)} = 4\zeta_1\zeta_2$$
$$N_2^{(e)} = \zeta_2(2\zeta_2 - 1) \qquad N_5^{(e)} = 4\zeta_2\zeta_3$$
$$N_3^{(e)} = \zeta_3(2\zeta_3 - 1) \qquad N_6^{(e)} = 4\zeta_3\zeta_1$$

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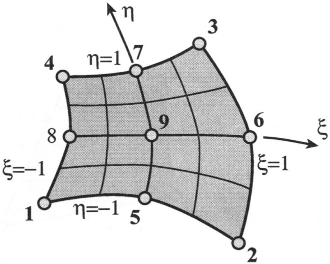
Terms in 1, ξ, η, ξ^2, $\xi\eta$, η^2

The biquadratic quad



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} & u_{x9} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} & u_{y9} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ \vdots \\ N_9^{(e)} \end{bmatrix}$$

$$N_1^{(e)} = \tfrac{1}{4}(1-\xi)(1-\eta)\xi\eta \qquad N_5^{(e)} = -\tfrac{1}{2}(1-\xi^2)(1-\eta)\eta$$

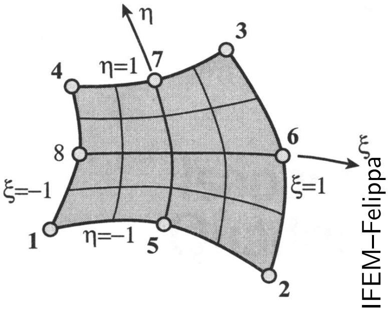
$$N_2^{(e)} = -\tfrac{1}{4}(1+\xi)(1-\eta)\xi\eta \qquad N_6^{(e)} = \tfrac{1}{2}(1+\xi)(1-\eta^2)\xi \qquad N_9^{(e)} = (1-\xi^2)(1-\eta^2)$$

$$\ldots \qquad \ldots$$

IFEM–Felippa

Terms in 1, ξ, η, ξ^2, $\xi\eta$, η^2, $\xi^2\eta$, $\xi\eta^2$, $\xi^2\eta^2$

The 8-node quad



- Corner nodes: $N_i = \dfrac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1)$
- Mid nodes, $\xi_i = 0$: $N_i = \dfrac{1}{2}(1-\xi^2)(1+\eta\eta_i)$
- Mid nodes, $\eta_i = 0$: $n_I = \dfrac{1}{2}(1-\eta^2)(1+\xi\xi_i)$

Terms in 1, ξ, η, ξ^2, $\xi\eta$, η^2, $\xi^2\eta$, $\xi\eta^2$

Approximated field

Polynomial basis

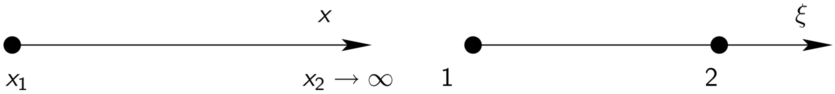
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & \xi & & \eta & & \\ & \xi^2 & & \xi\eta & & \eta^2 & \\ \xi^3 & & \xi^2\eta & & \xi\eta^2 & & \eta^3 \end{array}$$

Examples :

C2D4	$(1+\xi_i\xi)(1+\eta_i\eta)$
C2D8, corner	$0.25(-1+\xi_i\xi+\eta_i\eta)(1+\xi_i\xi)(1+\eta_i\eta)$
C2D8 middle	$0.25(1.-\xi^2)(1.+\eta_i\eta)$

The 2-node infinite element

Displacement is assumed to be q_1 at node 1 and $q_2 = 0$ at node 2



- Interpolation

$$N_1 = \frac{1-\xi}{2} \qquad N_2 = \frac{1+\xi}{2}$$

- Geometry

$$N_1^* \text{ such as } x = x_1 + \alpha \frac{1+\xi}{1-\xi} \qquad N_2^* = 0$$

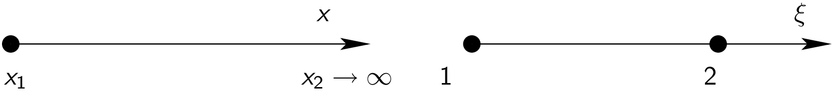
$$\xi = ?$$

- Resulting displacement interpolation

$$u(x) = ??$$

The 2-node infinite element

Displacement is assumed to be q_1 at node 1 and $q_2 = 0$ at node 2



- Interpolation

$$N_1 = \frac{1-\xi}{2} \qquad N_2 = \frac{1+\xi}{2}$$

- Geometry

$$N_1^* \text{ such as } x = x_1 + \alpha \frac{1+\xi}{1-\xi} \qquad N_2^* = 0$$

$$\xi = \frac{x - x_1 - \alpha}{x - x_1 + \alpha}$$

- Resulting displacement interpolation

$$u(x) = ?$$

The 2-node infinite element

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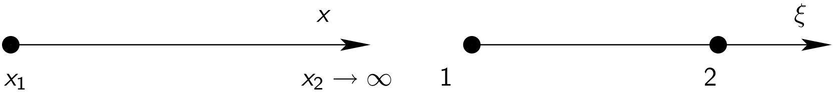


- Interpolation

$$N_1 = \frac{1-\xi}{2} \qquad N_2 = \frac{1+\xi}{2}$$

- Geometry

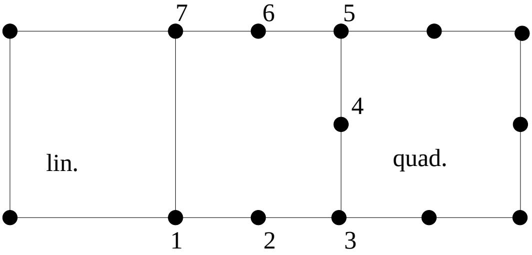
$$N_1^* \text{ such as } x = x_1 + \alpha \frac{1+\xi}{1-\xi} \qquad N_2^* = 0$$

$$\xi = \frac{x - x_1 - \alpha}{x - x_1 + \alpha}$$

- Resulting displacement interpolation

$$u(x) = N_1(x)\, q_1 = N_1(\xi(x))\, q_1 = \frac{\alpha q_1}{x - x_1 + \alpha}$$

Connecting element



Connection between a linear and a quadratic quad

- Quadratic interpolation with node number 8 in the middle of 1–7:

$$u(M) = N_1 q_1 + N_8 q_8 + N_7 q_7$$

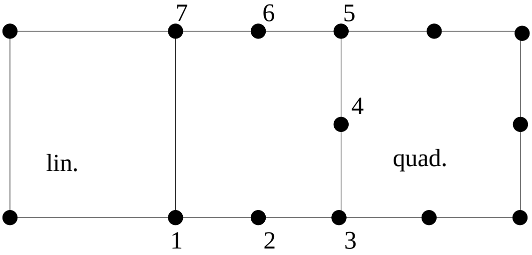
- On edge 1–7, in the linear element, the displacement should verify:

$$q_8 = ?$$

- Overloaded shape function in nodes 1 and 7 after suppressing node 8:

$$u(M) = ??$$

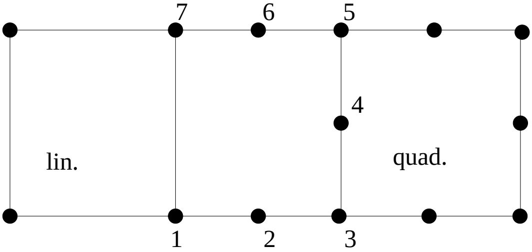
Connecting element



Connection between a linear and a quadratic quad

- Quadratic interpolation with node number 8 in the middle of 1–7:
$$u(M) = N_1 q_1 + N_8 q_8 + N_7 q_7$$
- On edge 1–7, in the linear element, the displacement should verify:
$$q_8 = (q_1 + q_7)/2$$
- Overloaded shape function in nodes 1 and 7 after suppressing node 8:
$$u(M) = ??$$

Connecting element



Connection between a linear and a quadratic quad

- Quadratic interpolation with node number 8 in the middle of 1–7:

$$u(M) = N_1 q_1 + N_8 q_8 + N_7 q_7$$

- On edge 1–7, in the linear element, the displacement should verify:

$$q_8 = (q_1 + q_7)/2$$

- Overloaded shape function in nodes 1 and 7 after suppressing node 8:

$$u(M) = \left(N_1 + \frac{N_8}{2}\right) q_1 + \left(N_7 + \frac{N_8}{2}\right) q_7$$

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Thermal conduction

Strong form:

$$\begin{array}{rrl} \text{"GIVEN} & r: & \Omega \rightarrow \mathbb{R}, \text{ a volumetric flux,} \\ & \phi^d: & \Gamma_f \rightarrow \mathbb{R}, \text{ a surface flux,} \\ & T^d: & \Gamma_u \rightarrow \mathbb{R}, \text{ a prescribed temperature,} \\ \text{FIND} & T: & \Omega \rightarrow \mathbb{R}, \text{ the temperature, such as:"} \end{array}$$

$$\begin{array}{lrl} \text{in } \Omega & \phi_{i,i} & = r \\ \text{on } \Gamma_u & T & = T^d \\ \text{on } \Gamma_F & -\phi_i n_i & = \Phi^d \end{array}$$

Constitutive equation (Fourier, flux (W/m^2) proportional to the temperature gradient)

$$\phi_i = -\kappa_{ij} T_{,j} \qquad \text{conductivity matrix:} \quad [\kappa] \quad (W/m.K)$$

Thermal conduction (2)

Weak form:

$\mathcal{S}$, trial solution space, such as $T = T^d$ on Γ_u

$\mathcal{V}$, variation space, such as $\delta T = 0$ on Γ_u

"GIVEN

$$\begin{aligned} r &: \quad \Omega \rightarrow \mathbb{R}, \text{ a volumetric flux,} \\ \Phi^d &: \quad \Gamma_f \rightarrow \mathbb{R}, \text{ a surface flux,} \\ T^d &: \quad \Gamma_u \rightarrow \mathbb{R}, \text{ a prescribed temperature,} \end{aligned}$$

FIND $T \in \mathcal{S}$ such as $\forall \delta T \in \mathcal{V}$

$$-\int_\Omega \phi_i \delta T_{,i} \, d\Omega = \int_\Omega \delta T r d\Omega + \int_{\Gamma_F} \delta T \Phi^d d\Gamma$$

"For any temperature variation compatible with prescribed temperature field around a state which respects equilibrium, the internal power variation is equal to the external power variation: $\delta T_{,i} \, \phi_i$ is in W/m^3"

T is present in $\phi_i = -\kappa_{ij} T_{,j}$

Elastostatic

Strong form:

$$\begin{array}{rcl} \text{volume } \Omega \text{ with prescribed volume forces } \underline{\mathbf{f}}^d & : & \sigma_{ij,j} + f_i = 0 \\ \text{surface } \Gamma_F \text{ with prescribed forces } \underline{\mathbf{F}}^d & : & F_i^d = \sigma_{ij} n_j \\ \text{surface } \Gamma_u \text{ with prescribed displacements } \underline{\mathbf{u}}^d & : & u_i = u_i^d \end{array}$$

Constitutive equation: $\sigma_{ij} = \Lambda_{ijkl}\varepsilon_{kl} = \Lambda_{ijkl} u_{k,l}$

So that: $\Lambda_{ijkl} u_{k,lj} + f_i = 0$

Principle of virtual power

Weak form:

$$\begin{array}{rcl} \text{volume } V \text{ with prescribed volume forces} & : & \underline{\mathbf{f}}^d \\ \text{surface } \Gamma_F \text{ with prescribed forces} & : & \underline{\mathbf{F}}^d \\ \text{surface } \Gamma_u \text{ with prescribed displacements} & : & \underline{\mathbf{u}}^d \end{array}$$

Virtual displacement rate $\dot{\underline{\mathbf{u}}}$ kinematically admissible ($\dot{\underline{\mathbf{u}}} = \dot{\underline{\mathbf{u}}}^d$ on Γ_u)

The variation $\dot{\underline{\mathbf{u}}}$ is such as: $\dot{\underline{\mathbf{u}}} = 0$ on Γ_u. Galerkin form writes, $\forall \dot{\underline{\mathbf{u}}}$:

$$\int_\Omega \underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}} \, d\Omega = \int_\Omega \underline{\mathbf{f}}^d \; \dot{\underline{\mathbf{u}}} \, d\Omega + \int_{\Gamma_F} \underline{\mathbf{F}}^d \; \dot{\underline{\mathbf{u}}} \, dS$$

Discrete form of virtual power

Application of Galerkin approach for continuum mechanics:

$$\text{virtual displacement rate } \underline{\dot{\mathbf{u}}} \equiv w^h \quad ; \quad \underset{\sim}{\sigma} \equiv u^h{}_{,x}$$

$\{\dot{u}^e\}$, nodal displacements allow us to compute $\underline{\dot{\mathbf{u}}}$ and $\underset{\sim}{\dot{\varepsilon}}$:

$$\underline{\dot{\mathbf{u}}} = [N]\{\dot{u}^e\} \quad ; \quad \underset{\sim}{\dot{\varepsilon}} = [B]\{\dot{u}^e\}$$

Galerkin form writes, $\forall\{\dot{u}^e\}$:

$$\sum_{elt} \left(\int_\Omega \{\sigma(\{u^e\}).[B].\{\dot{u}^e\} \; d\Omega \right) = \sum_{elt} \Bigg(\int_\Omega \underline{\mathbf{f}}^d \; .[N].\{\dot{u}^e\} \, d\Omega + \int_{\Gamma_F} \underline{\mathbf{F}}^d \; .[N].\{\dot{u}^e\} \, dS \Bigg)$$

Internal and external forces

In each element e:

- Internal forces:

$$\{F^e_{int}\} = \int_\Omega \{\sigma(\{u^e\}).[B]\, d\Omega = \int_\Omega [B]^T \{\sigma(\{u^e\})\, d\Omega$$

- External forces:

$$\{F^e_{ext}\} = \int_\Omega \underline{\mathbf{f}}^d .[N] d\Omega + \int_{\Gamma_F} \underline{\mathbf{F}}^d .[N] dS$$

The solution of the problem: $\{F^e_{int}(\{u^e\})\} = \{F^e_{ext}\}$ with Newton iterative algorithm will use the jacobian matrix :

$$\begin{aligned}
[K^e] &= \frac{\partial\{F^e_{int}\}}{\partial\{u^e\}} \\
&= \int_\Omega [B]^T . \frac{\partial\{\sigma\}}{\partial\{\varepsilon\}} . \frac{\partial\{\varepsilon\}}{\partial\{u^e\}}\, d\Omega \\
&= \int_\Omega [B]^T . \frac{\partial\{\sigma\}}{\partial\{\varepsilon\}} .[B]\, d\Omega
\end{aligned}$$

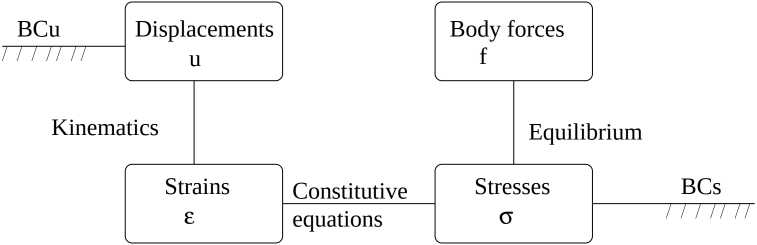
Linear and non linear behavior

- Applying the principle of virtual power $\equiv$ Stationnary point of Potential Energy
- For elastic behavior

$$[K^e] = \int_\Omega [B]^T . [\underset{\approx}{\mathbf{\Lambda}}] . [B] \, d\Omega$$

  is symmetric, positive definite (true since $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$ are conjugated)
- For non linear behavior, one has to examine $[L_c] = \left[\dfrac{\partial\{\sigma\}}{\partial\{\varepsilon\}}\right]$. Note that $[L_c]$ can be approached (quasi-Newton).
- $\{F^e_{ext}\}$ may depend on $\{u^e\}$ (large displacements).

Elastostatic, strong and weak form, a summary



STRONG

- BCu: $u = u^d$ on Γ_u
- Kinematics: $\varepsilon = [B]\, u$ in Ω
- Constitutive equation: $\sigma = \Lambda\varepsilon$
- Equilibrium: $[B]\,\sigma + \mathrm{f} = 0$
- BCs: $\sigma n = F$ on Γ_F

WEAK

- BCu: $u^h = u^d$ on Γ_u
- Kinematics: $\varepsilon = [B]\, u^h$ in Ω
- Constitutive equation: $\sigma = \Lambda\varepsilon$
- Equilibrium: $\delta\Pi = 0$
- BCs: $\delta\Pi = 0$

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Matrix–vectors formulation of the weak form of the problem

$$[K]\{q\} = \{F\}$$

- Thermal conduction:

$$[K] = \int_{\Omega} [B]^T [\kappa] [B] \, d\Omega \qquad \{F\} = \int_{\Omega} [N] \, r d\Omega + \int_{\partial\Omega} [N] \, \Phi^d d\Gamma$$

- Elasticity:

$$[K] = \int_{\Omega} [B]^T [\Lambda] [B] \, d\Omega \qquad \{F\} = \int_{\Omega} [N] \, \underline{\mathbf{f}}^d d\Omega + \int_{\partial\Omega} [N] \, \underline{\mathbf{F}}^d d\Gamma$$

In each element e:

- Internal forces:

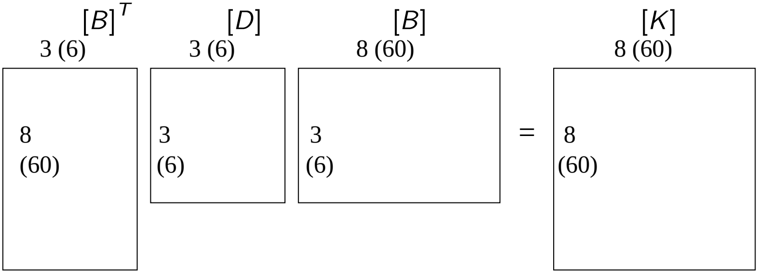
$$\{F^e_{int}\} = \int_{\Omega} \{\sigma(\{u^e\}).[B] \, d\Omega = \int_{\Omega} [B]^T \{\sigma(\{u^e\}) \, d\Omega$$

- External forces:

$$\{F^e_{ext}\} = \int_{\Omega} \underline{\mathbf{f}}^d .[N] d\Omega + \int_{\Gamma_F} \underline{\mathbf{F}}^d .[N] dS$$

The stiffness matrix

Example of a 4-node quad and of a 20-node hexahedron ()

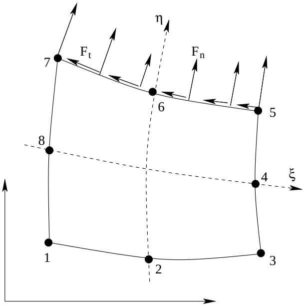


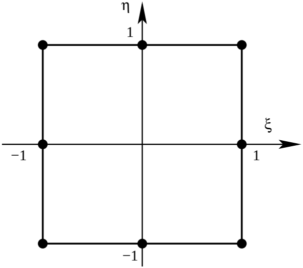
The element stiffness matrix is a square matrix, symmetric, with no zero inside.

Its size is equal to the number of dof of the element.

Nodal forces (1)

$$\{F^e_{ext}\} = \int_{\Gamma_F} [N]^T \, \underline{F}^d dS$$





$$F_x ds = F_t dx - F_n dy$$
$$F_y ds = F_n dx + F_t dy$$

with $\begin{pmatrix} dx \\ dy \end{pmatrix} = [J] \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}$

Nodal forces (2)

Integration on edge 5–7: $\quad dx = \dfrac{\partial x}{\partial \xi}\, d\xi \qquad dy = \dfrac{\partial y}{\partial \xi}\, d\xi$

Components 9, 10, for the nodes 5; 11, 12 for nodes 6; 13, 14 for nodes 7

$$F^e_{ext}(2i-1) = e \int_{-1}^{1} N_i \left(F_t \frac{\partial x}{\partial \xi} - F_n \frac{\partial y}{\partial \xi} \right) d\xi$$

$$F^e_{ext}(2i) = e \int_{-1}^{1} N_i \left(F_n \frac{\partial x}{\partial \xi} + F_t \frac{\partial y}{\partial \xi} \right) d\xi$$

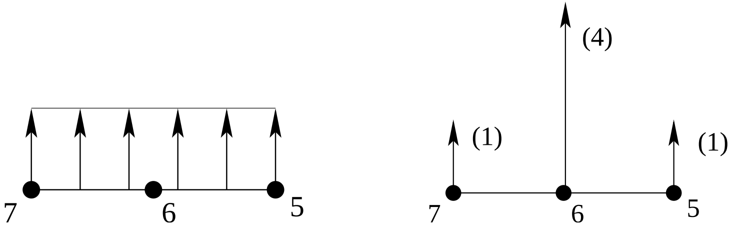
Example, for a pressure $F_n = p$, and no shear $(F_t = 0)$ on the 5–7 edge of a 8-node rectangle

$$-a \leqslant x \leqslant a \quad y = b \qquad \text{represented by } -1 \leqslant \xi \leqslant 1 \quad \eta = 1$$

$$\frac{\partial x}{\partial \xi} = a \qquad \frac{\partial y}{\partial \xi} = 0$$

$$N_5 = \xi(1+\xi)/2 \quad N_6 = 1-\xi^2 \quad N_7 = -\xi(1-\xi)/2$$

Nodal forces (3)

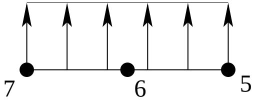


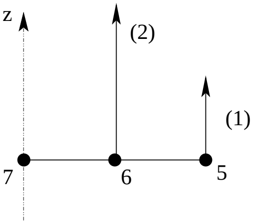
$$F_{10} = F_{5y} = e\int_{-1}^{1} \frac{1}{2}\xi(1+\xi)pad\xi = \frac{ap}{3}$$
$$F_{12} = F_{6y} = e\int_{-1}^{1} (1-\xi^2)pad\xi = \frac{4ap}{3}$$

The nodal forces at the middle node are 4 times the nodal forces at corner nodes for an uniform pressure (distribution 1–2–1–2–1... after adding the contribution of each element)

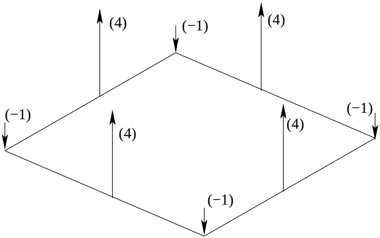
Nodal forces (4)

- Axisymmetric 8-node quad





- Face of a 20-node hexahedron

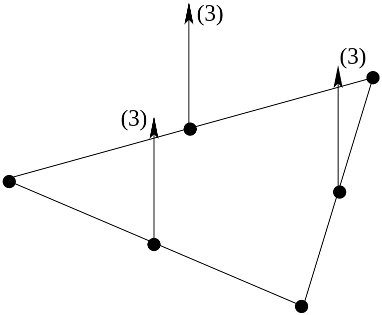


Nodal forces (5)

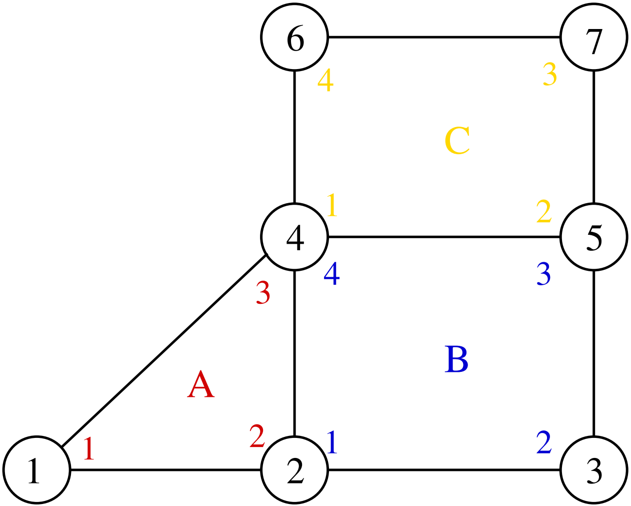
- Face of a 27-node hexahedron

who knows ?

- Face of a 15-node hexahedron



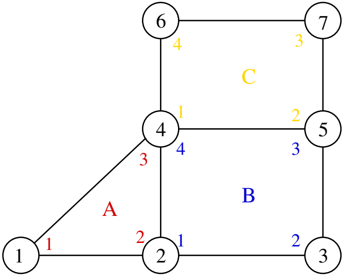
Assembling the global matrix



Local versus *global numbering*

Assembling the global matrix

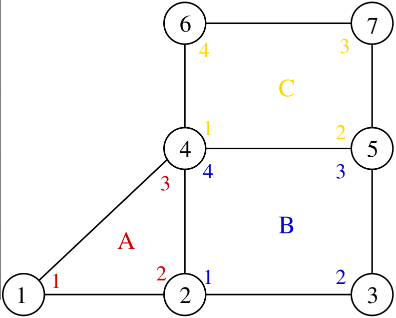
$$\begin{pmatrix} F_1 & = & F_1^A & & \\ F_2 & = & F_2^A & +F_1^B & \\ F_3 & = & & +F_2^B & \\ F_4 & = & F_3^A & +F_4^B & +F_1^C \\ F_5 & = & & +F_3^B & +F_2^C \\ F_6 & = & & & +F_4^C \\ F_7 & = & & & +F_3^C \end{pmatrix} \qquad \begin{pmatrix} q_1 & = & q_1^A & & \\ q_2 & = & q_2^A & = q_1^B & \\ q_3 & = & & = q_2^B & \\ q_4 & = & q_3^A & = q_4^B & = q_1^C \\ q_5 & = & & = q_3^B & = q_2^C \\ q_6 & = & & & = q_4^C \\ q_7 & = & & & = q_3^C \end{pmatrix}$$



Assembling the global matrix

11						

1



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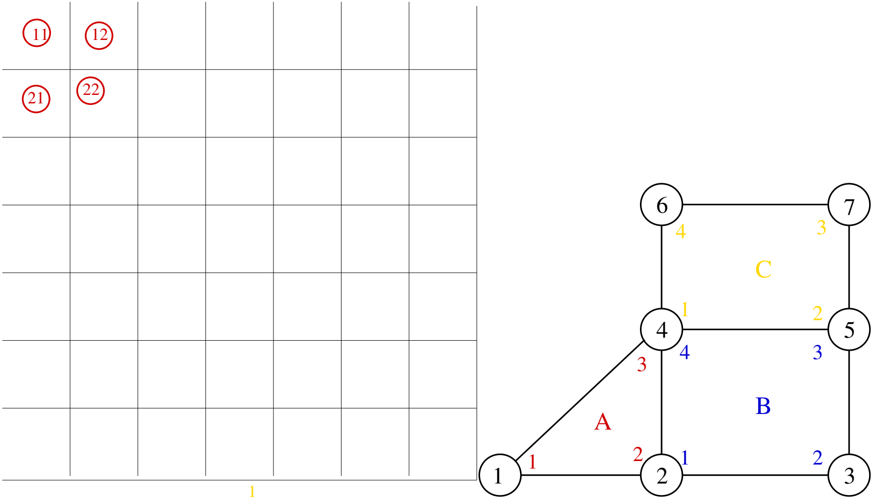

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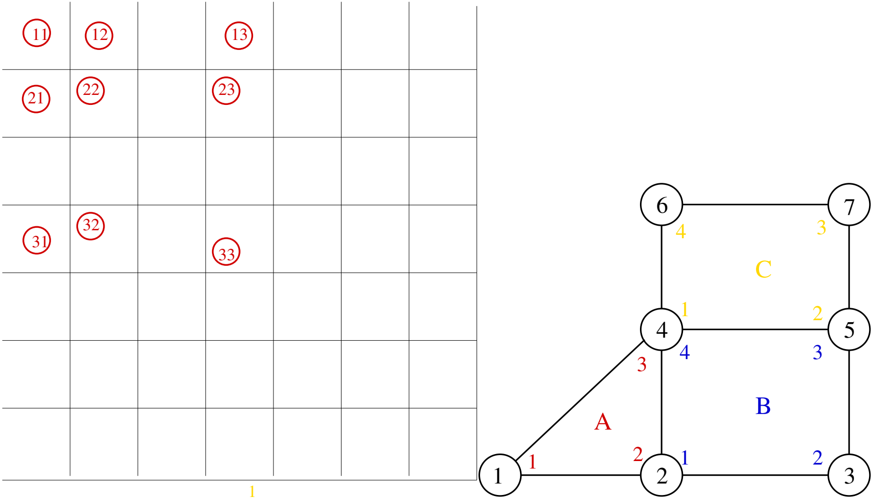

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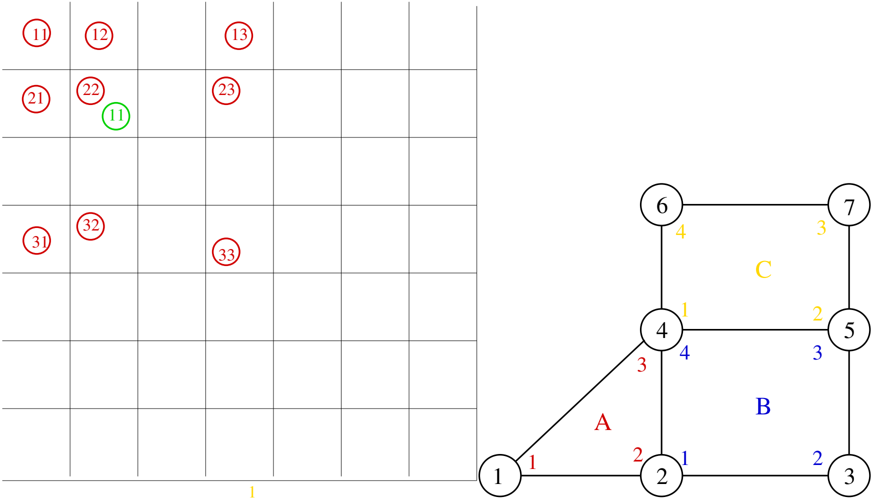

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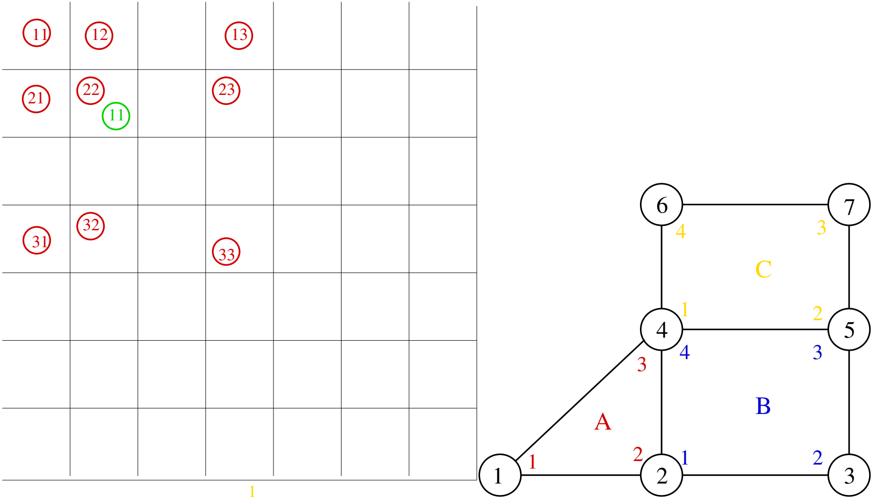

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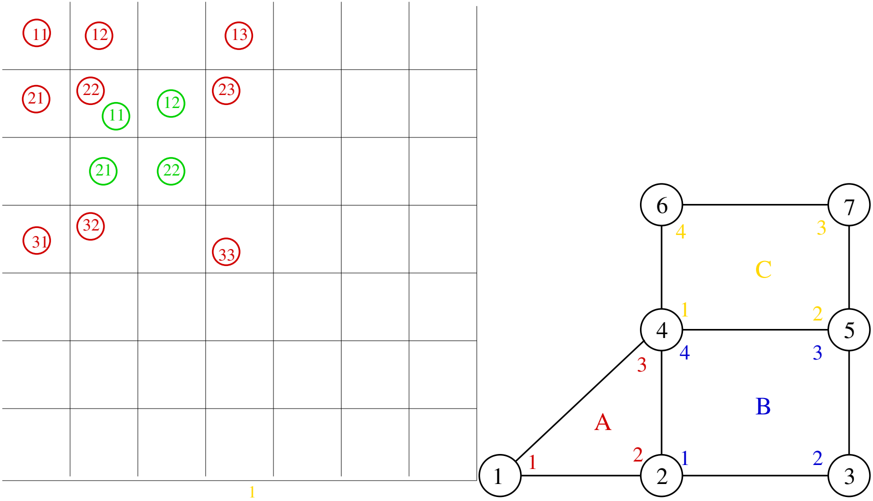

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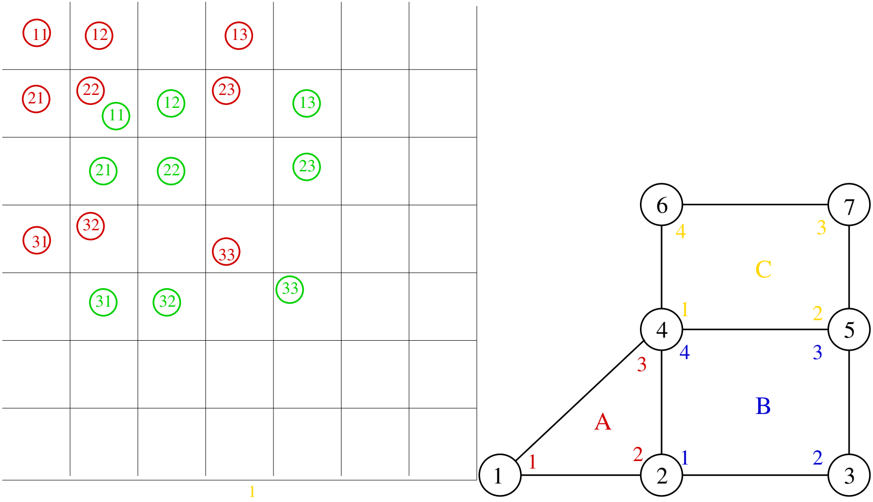

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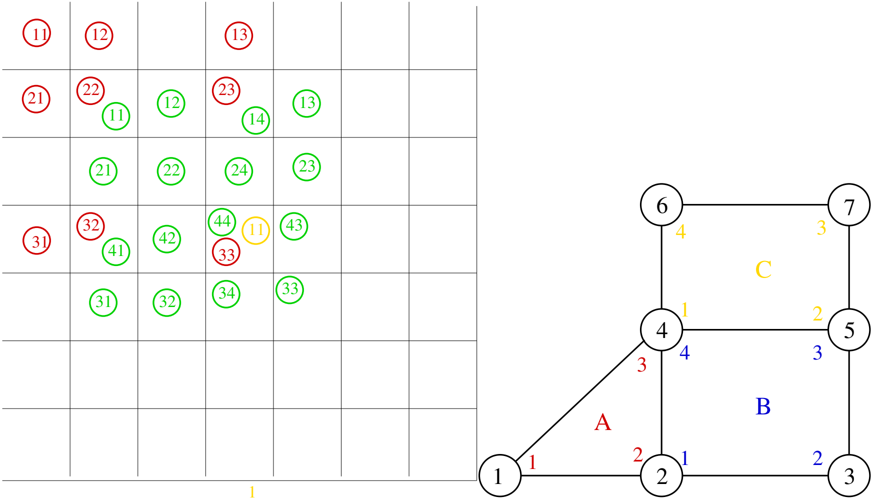

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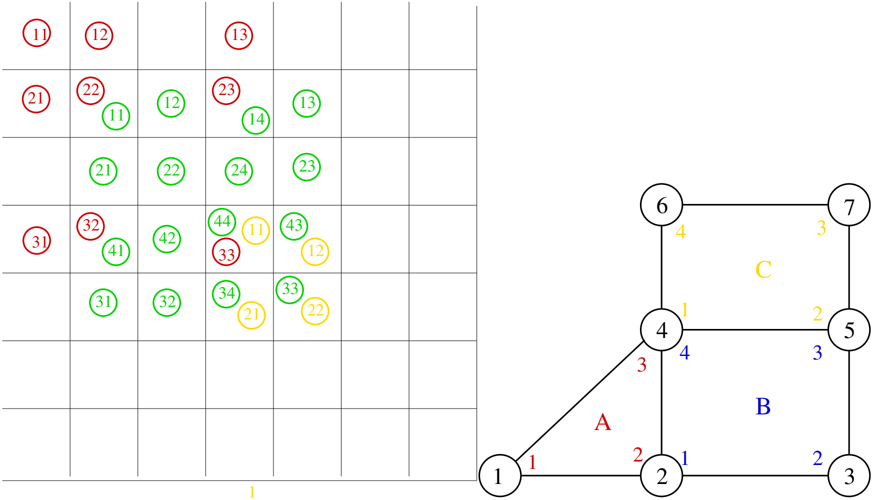

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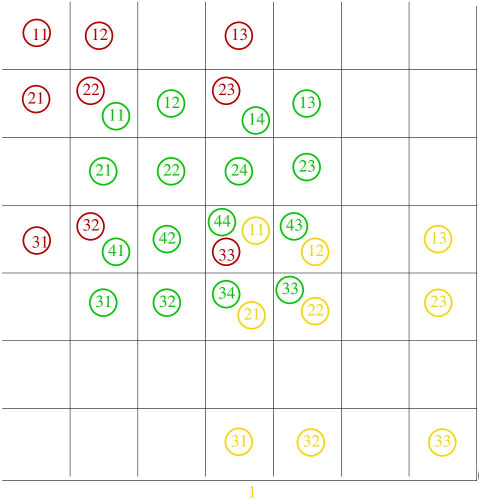

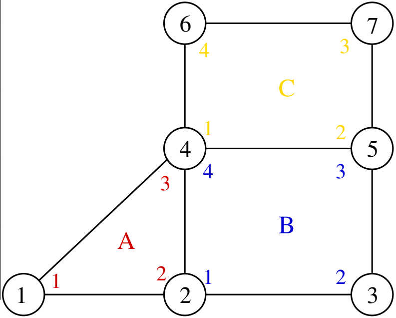

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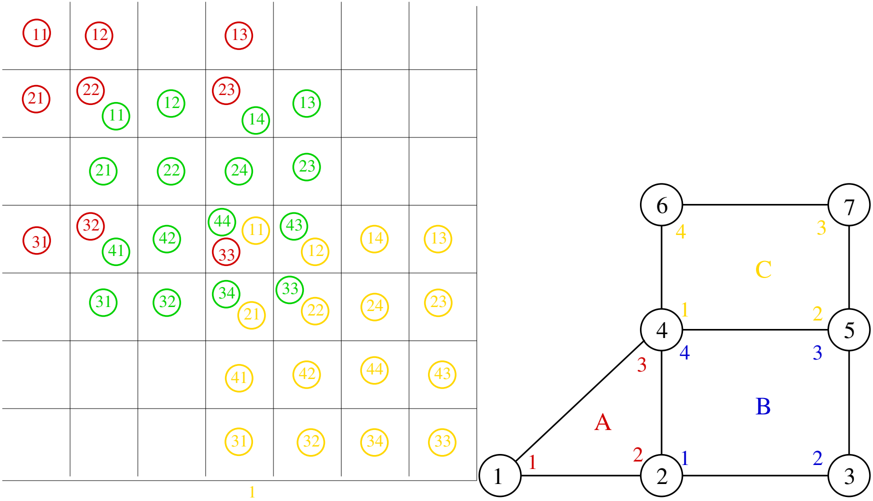

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Global algorithm

For each loading increment, do while $\|\{R\}_{iter}\| > EPSI$:
$iter = 0; iter < ITERMAX; iter++$

1. Update displacements: $\Delta\{u\}_{iter+1} = \Delta\{u\}_{iter} + \delta\{u\}_{iter}$
2. Compute $\Delta\{\varepsilon\} = [B].\Delta\{u\}_{iter+1}$ then $\Delta\underset{\sim}{\varepsilon}$ for each Gauss point
3. Integrate the constitutive equation: $\Delta\underset{\sim}{\varepsilon} \rightarrow \Delta\underset{\sim}{\sigma}, \Delta\alpha_I, \dfrac{\Delta\underset{\sim}{\sigma}}{\Delta\underset{\sim}{\varepsilon}}$
4. Compute int and ext forces: $\{F_{int}(\{u\}_t + \Delta\{u\}_{iter+1})\}, \{F_e\}$
5. Compute the residual force: $\{R\}_{iter+1} = \{F_{int}\} - \{F_e\}$
6. New displacement increment: $\delta\{u\}_{iter+1} = -[K]^{-1}.\{R\}_{iter+1}$

Convergence

- Value of the residual forces $< R_\epsilon$, e.g.

$$||\{R\}||_n = \left(\sum_i R_i^n\right)^{1/n} \quad ; \quad ||\{R\}||_\infty = \max_i |R_i|$$

- Relative values:

$$\frac{||\{R\}_i - \{R\}_e||}{||\{R\}_e||} < \epsilon$$

- Displacements

$$\left|\left|\{U\}_{k+1} - \{U\}_k\right|\right|_n < U_\epsilon$$

- Energy

$$\left[\{U\}_{k+1} - \{U\}_k\right]^T . \{R\}_k < W_\epsilon$$