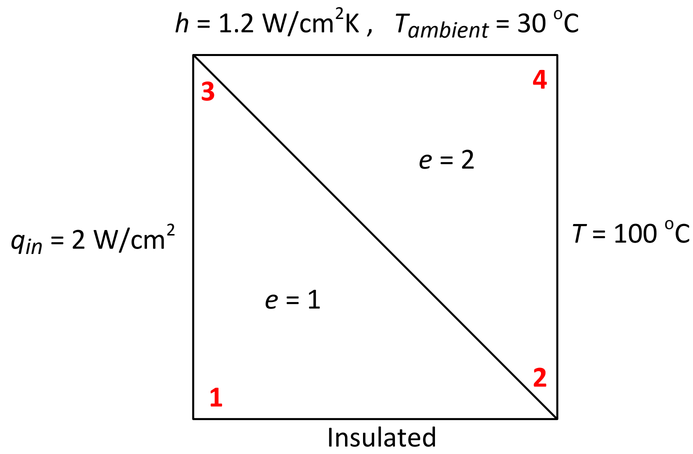
2D FEM Hand Solutions

Example 1: Consider steady heat conduction on the following two-dimensional square plate of size 5 cm x 5 cm. The plate has a constant thermal conductivity of 2 W/cmK. Bottom side is insulated. Right side is kept at a constant temperature of 100 ^oC. Top side is subject to convective heat transfer. A constant heat flux is specified to be coming into the plate from the left side.



Using two 3-node triangular elements, calculate the temperature distribution on the plate.

Solution:

The governing equation for steady, heat conduction, without heat generation is

$$-\nabla \cdot (k\, \nabla \mathrm{T}) = 0$$

$$-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$

After forming the weighted residual statement and applying integration by parts we get the following weak form

$$0 = \int_{\Omega}\left(\frac{\partial w}{\partial x}k\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}k\frac{\partial T}{\partial y}\right)dxdy \; - \; \oint_{\Gamma} w\left(n_x\, k\frac{\partial T}{\partial x} + n_y\, k\frac{\partial T}{\partial y}\right)ds$$

where temperature, *T*, is the primary variable and heat flux, $q_n = n_x\, k\frac{\partial T}{\partial x} + n_y\, k\frac{\partial T}{\partial y}$, is the secondary variable.

Using Galerkin formulation the elemental system can be written as

$$[K^e]\{T^e\} = \{F^e\} + \{Q^e\}$$

where

$$K_{ij}^e = \int_{\Omega^e} k \left(\frac{\partial S_i}{\partial x} \frac{\partial S_j}{\partial x} + \frac{\partial S_i}{\partial y} \frac{\partial S_j}{\partial y} \right) dxdy$$

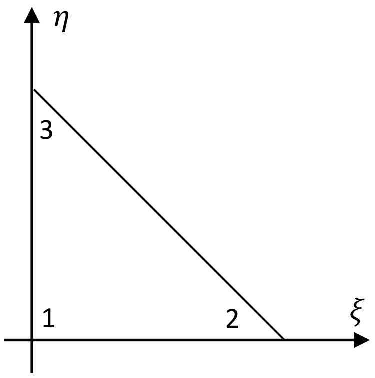
$$F_i^e = 0$$

$$Q_i^e = \oint_\Gamma S_i \, q_n \, ds$$

It is easier to work with the shape functions defined on the master element as functions of ξ and η. Elemental stiffness matrix written using master element coordinates is

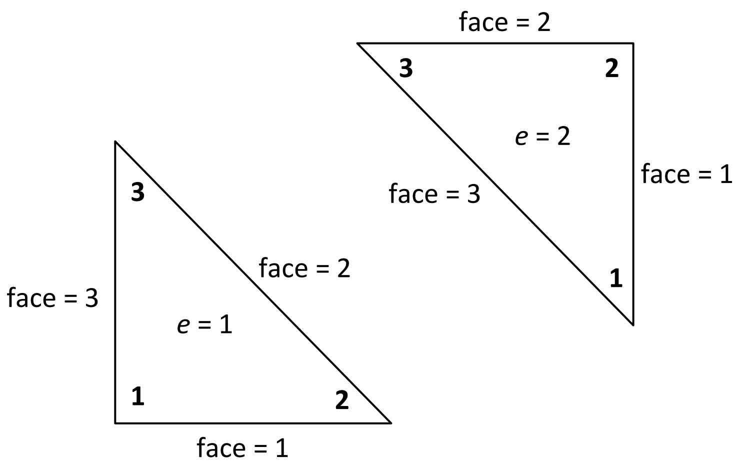
$$K_{ij}^e = \int_{\Omega^e} k \left[\left(\frac{\partial S_i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial S_i}{\partial \eta} \frac{\partial \eta}{\partial x} \right) \left(\frac{\partial S_j}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial S_j}{\partial \eta} \frac{\partial \eta}{\partial x} \right) + \left(\frac{\partial S_i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial S_i}{\partial \eta} \frac{\partial \eta}{\partial y} \right) \left(\frac{\partial S_j}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial S_j}{\partial \eta} \frac{\partial \eta}{\partial y} \right) \right] |J^e| \, d\xi d\eta$$

For 3-node triangular element shape functions are



$$S_1 = 1 - \xi - \eta \ , \quad S_2 = \xi \ , \quad S_3 = \eta$$

We also need to calculate the Jacobian matrix, its determinant and inverse for each element. To do this we first need to select local node numbering for each element. We'll use the following local node numbering



IMPORTANT: We numbered the nodes of each element in a CCW order, which is also the order used in numbering the nodes of the master element. We need to do it this way in order to calculate Jacobian matrix and its determinant properly. The selection of the first node of each element is arbitrary though.

IMPORTANT: Element face numbering rule that we use is : Face 1 is the face between nodes 1 and 2, face 2 is the face between nodes 2 and 3, face 3 is between nodes 3 and 1.

Jacobian calculation for the first element:

$$[J^1] = \begin{bmatrix} \frac{\partial S_1}{\partial \xi} & \frac{\partial S_2}{\partial \xi} & \frac{\partial S_3}{\partial \xi} \\ \frac{\partial S_1}{\partial \eta} & \frac{\partial S_2}{\partial \eta} & \frac{\partial S_3}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1^1 & y_1^1 \\ x_2^1 & y_2^1 \\ x_3^1 & y_3^1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0.05 & 0 \\ 0 & 0.05 \end{bmatrix} = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}$$

$$|J^1| = 2.5 \times 10^{-3} \quad , \quad [J^1]^{-1} = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$$

Jacobian calculation for the second element:

$$[J^1] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.05 & 0 \\ 0.05 & 0.05 \\ 0 & 0.05 \end{bmatrix} = \begin{bmatrix} 0 & 0.05 \\ -0.05 & 0.05 \end{bmatrix}$$

$$|J^1| = 2.5 \times 10^{-3} \quad , \quad [J^1]^{-1} = \begin{bmatrix} 20 & -20 \\ 20 & 0 \end{bmatrix}$$

Remembering that the components of the inverse of the Jacobian matrix are

$$[J]^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}$$

entries of the stiffness matrix of the first element can be calculated as

$$K_{11}^1 = \int_{\xi=0}^{1} \int_{\eta=0}^{1-\xi} 200\{[(-1)(20) + (-1)(0)]\,[(-1)(20) + (-1)(0)] + [(-1)(0) + (-1)(20)]\,[(-1)(0) + (-1)(20)]\}\, 2.5 \times 10^{-3} \; d\xi \, d\eta$$

$K_{11}^1 = 200$ (You can use Gauss Quadrature integration or evaluate it analytically)

IMPORTANT : Be careful how the limits of the integral are set. The limits will be from -1 to 1 for a quadrilateral master element, but not for a triangular one.

Calculating the remaining entries, stiffness matrix of the first element becomes

$$[K^1] = \begin{bmatrix} 200 & -100 & -100 \\ -100 & 100 & 0 \\ -100 & 0 & 100 \end{bmatrix}$$

Note that this is a symmetric matrix, which is expected because $K_{ij}^1 = K_{ji}^1$

Stiffness matrix for the second element can be calculated as follows (which is again symmetric)

$$[K^2] = \begin{bmatrix} 100 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 100 \end{bmatrix}$$

IMPORTANT : Although stiffness matrix integral does not depend on *x* or *y* and the shape and size of both elements are similar, $[K^1]$ is not equal to $[K^2]$. This is because of our local node numbering selection. Will the elemental stiffness matrices be the same if we change the local node numbering of the second element such that we start from the corner with the right angle and number the nodes in CCW?

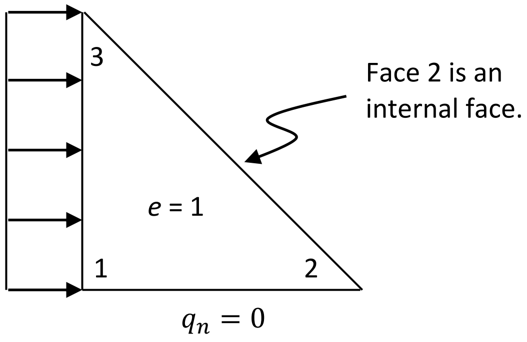
In order to calculate $\{Q\}$ vectors of the elements we need to examine the given BCs. At the bottom, top and left boundaries of the domain, natural and mixed type BCs are provided and the $\{Q\}$ vectors of the elements will get contribution from the faces located at these boundaries.

For element 1:

$$q_n = n_x k \frac{\partial T}{\partial x} + n_y k \frac{\partial T}{\partial y}$$

$$q_n = -k \frac{\partial T}{\partial x}$$

$$q_n = 2 \times 10^4 \; W/m^2$$



For the left boundary $n_x = -1$ and $n_y = 0$. It is also given in the problem statement that heat is coming into the domain, i.e. $\partial T / \partial x < 0$. Therefore q_n has a positive value on this boundary.

$q_n = 0$ at the bottom boundary since it is insulated. Face 2, located between local nodes 2 and 3 is an internal face (not located on a real boundary) and the boundary integral for that face does not need to be calculated.

$$\{Q^1\} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \end{Bmatrix}$$

Let's start with Q_1^1.

$$\begin{aligned} Q_1^1 = \oint_{\Gamma^1} S_1\, q_n\, ds &= \int_{\text{face 1}} S_1\, q_n\, ds + \int_{\text{face 2}} S_1\, q_n\, ds + \int_{\text{face 3}} S_1\, q_n\, ds \\ &= Q_1^1|_{\text{face 1}} + Q_1^1|_{\text{face 2}} + Q_1^1|_{\text{face 3}} \\ &= 0 + 0 + Q_1^1|_{\text{face 3}} \end{aligned}$$

IMPORTANT :

- $Q_1^1|_{\text{face 1}}$ is zero because $q_n = 0$ on face 1.
- $Q_1^1|_{\text{face 2}}$ is zero because S_1 is zero on face 2 (Node 1 does not lie on face 2).
- $Q_1^1|_{\text{face 3}}$ should be calculated.

Now let's study at Q_2^1. Since this node lies on an EBC boundary the PV is known at this node. Therefore we do not need to do any SV calculation for this node at all.

Finally let's look at Q_3^1.

$$\begin{aligned} Q_3^1 = \oint_{\Gamma^1} S_3\, q_n\, ds &= \int_{\text{face 1}} S_3\, q_n\, ds + \int_{\text{face 2}} S_3\, q_n\, ds + \int_{\text{face 3}} S_3\, q_n\, ds \\ &= Q_3^1|_{\text{face 1}} + Q_3^1|_{\text{face 2}} + Q_3^1|_{\text{face 3}} \\ &= 0 + Q_2^1|_{\text{face 2}} + 0 \end{aligned}$$

IMPORTANT :

- $Q_2^1|_{\text{face 1}}$ is zero because face 1 can not contribute to Q_3^1.
- $Q_3^1|_{\text{face 2}}$ does NOT need to be calculated because face 2 is an internal face.
- $Q_3^1|_{\text{face 3}}$ needs to be calculated.

To summarize for element 1 we only need to calculate $Q_1^1|_{\text{face 3}}$ and $Q_3^1|_{\text{face 3}}$.

On face 3 a constant SV is specified and we studied this case in our lectures. Face 3 joins nodes 1 and 3 and the boundary integral over this face will provide the following contributions

$$Q_1^1|_{\text{face 3}} = \frac{q_n L_{13}}{2} = 500 \qquad \text{and} \qquad Q_3^1|_{\text{face 3}} = \frac{q_n L_{13}}{2} = 500$$

Therefore the $\{Q\}$ vector of the first element will be

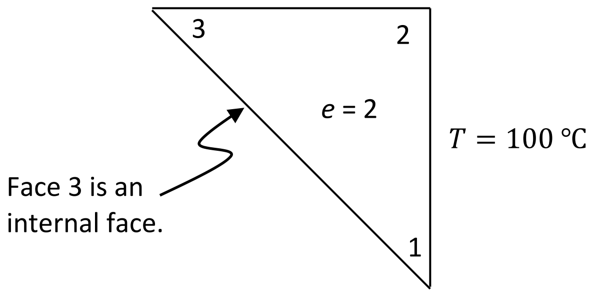
$$\{Q^1\} = \begin{Bmatrix} 500 \\ Q_2^1 \\ 500 + Q_3^1|_{\text{face 2}} \end{Bmatrix}$$

To remind again, Q_2^1 is not calculated because of the specified EBC at this node and $Q_3^1|_{\text{face 2}}$ is not calculated because face 2 is internal and this value will cancel out during the assembly.

<u>Q calculation for element 2:</u>

$$q_n = k\frac{\partial T}{\partial y} = -h(T - T_{amb})$$

$$= -1.2 \times 10^4\ (T - 30)$$



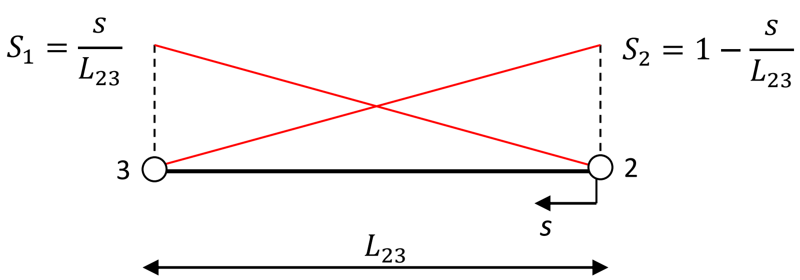
IMPORTANT : For the second element face 1 has EBC specified therefore no SV calculation for the nodes of this face, i.e. we do not need to calculate Q_1^2 and Q_2^2. Face 3 is an internal face and its contribution will cancel out during the assembly. Face 2 has mixed type BC and its contribution the Q values should be evaluated.

IMPORTANT : Make sure that you understand $k\frac{\partial T}{\partial y} = -h(T - T_{amb})$ equation, which has nothing to do with our FEM formulation, but simply states the balance of conductive and convective heat transfer at the top boundary.

Let's write the mixed type BC of face 2 as follows

$$q_n = \alpha T + \beta \qquad \text{where} \qquad \alpha = -1.2 \times 10^4 \qquad \text{and} \qquad \beta = 3.6 \times 10^5$$

Part of the boundary integral over face 2 will contribute to Q_2^2 and Q_3^2 since this face is between nodes 2 and 3. These contributions can be calculated similar to the way we calculated the integrals for a NBC with constant q_n. Consider 1D shape functions over face 2



$$Q_2^2|_{\text{face 2}} = \int_{s=0}^{L_{23}} S_2\, q_n\, ds \; = \; \int_0^{L_{23}} \left(1 - \frac{s}{L_{23}}\right)(\alpha T + \beta)\, ds$$

Temperature that appears in the integral is the temperature distribution over face 2 of element 2. It can be represented using the 1D shape functions and nodal temperature values as follows

$$T = \left(1 - \frac{s}{L_{23}}\right) T_2^2 \; + \; \left(\frac{s}{L_{23}}\right) T_3^2$$

Using this in the above integral we get

$$Q_2^2|_{\text{face 2}} = \int_0^{L_{23}} \left(1 - \frac{s}{L_{23}}\right) \left[\alpha \left(1 - \frac{s}{L_{23}}\right) T_2^2 \; + \; \alpha \left(\frac{s}{L_{23}}\right) T_3^2 + \beta\right] ds$$

Evaluating this integral

$$Q_2^2|_{\text{face 2}} \; = \; \frac{\beta}{2} L_{23} \; + \; \alpha \frac{L_{23}}{3} T_2^2 \; + \; \alpha \frac{L_{23}}{6} T_3^2$$

$$Q_2^2|_{\text{face 2}} \; = \; 9000 \; - \; 200\, T_2^2 \; - \; 100\, T_3^2$$

IMPORTANT: As stated previously Q_2^2 is not necessary for us due to the specified BC. Therefore the above calculation is unnecessary, but provided for demonstration purposes. The calculated value will not be used in the rest of this document.

Similarly we need to evaluate the contribution of the boundary integral over face 2 of the second element to node 3 of the second element.

$$Q_3^2|_{\text{face 2}} = \int_{s=0}^{L_{23}} S_3\, q_n\, ds \; = \; \int_0^{L_{23}} \left(\frac{s}{L_{23}}\right) \left[\alpha \left(1 - \frac{s}{L_{23}}\right) T_2^2 \; + \; \alpha \left(\frac{s}{L_{23}}\right) T_3^2 + \beta\right] ds$$

$$Q_3^2|_{\text{face 2}} \; = \; \frac{\beta}{2} L_{23} \; + \; \alpha \frac{L_{23}}{6} T_2^2 \; + \; \alpha \frac{L_{23}}{3} T_3^2$$

$$Q_3^2|_{\text{face 2}} \; = \; 9000 \; - \; 100\, T_2^2 \; - \; 200\, T_3^2$$

Therefore $\{Q\}$ vector of the second element will be

$$\{Q^2\} = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ 9000 \; - \; 100\, T_2^2 \; - \; 200\, T_3^2 \; + \; Q_3^2|_{\text{face 3}} \end{Bmatrix}$$

Now we can assemble the elemental systems to get the global system.

<u>Elemental system for the first element is:</u>

$$\begin{bmatrix} 200 & -100 & -100 \\ -100 & 100 & 0 \\ -100 & 0 & 100 \end{bmatrix} \begin{Bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 500 \\ Q_2^1 \\ 500 + Q_3^1|_{\text{face 2}} \end{Bmatrix}$$

<u>Elemental system for the second element is:</u>

$$\begin{bmatrix} 100 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 100 \end{bmatrix} \begin{Bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ 9000 - 100\,T_2^2 - 200\,T_3^2 + Q_3^2|_{\text{face 3}} \end{Bmatrix}$$

To assemble these systems we need to use the following LtoG matrix

$$LtoG = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

Assembled global system becomes

$$\begin{bmatrix} 200 & -100 & -100 & 0 \\ -100 & 100 + 100 & 0 + 0 & -100 \\ -100 & 0 + 0 & 100 + 100 & -100 \\ 0 & -100 & -100 & 200 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 500 \\ Q_2^1 + Q_2^2 \\ 500 + Q_3^1|_{\text{face 2}} + 9000 - 100\,T_2^2 - 200\,T_3^2 + Q_3^2|_{\text{face 3}} \\ Q_2^2 \end{Bmatrix}$$

Due to the balance of secondary variables, the following sum should be zero

$$Q_3^1|_{\text{face 2}} + Q_3^2|_{\text{face 3}} = 0$$

Simplified assembled system is

$$\begin{bmatrix} 200 & -100 & -100 & 0 \\ -100 & 200 & 0 & -100 \\ -100 & 0 & 200 & -100 \\ 0 & -100 & -100 & 200 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 500 \\ Q_2 \\ 500 + 9000 - 100\,T_2^2 - 200\,T_3^2 \\ Q_4 \end{Bmatrix}$$

Let's transfer the temperature values inside $\{Q\}$ to the stiffness matrix (Note that $T_2^2 = T_4$ and $T_3^2 = T_3$).

$$\begin{bmatrix} 200 & -100 & -100 & 0 \\ -100 & 200 & 0 & -100 \\ -100 & 0 & 400 & 0 \\ 0 & -100 & -100 & 200 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 500 \\ Q_2 \\ 9500 \\ Q_4 \end{Bmatrix}$$

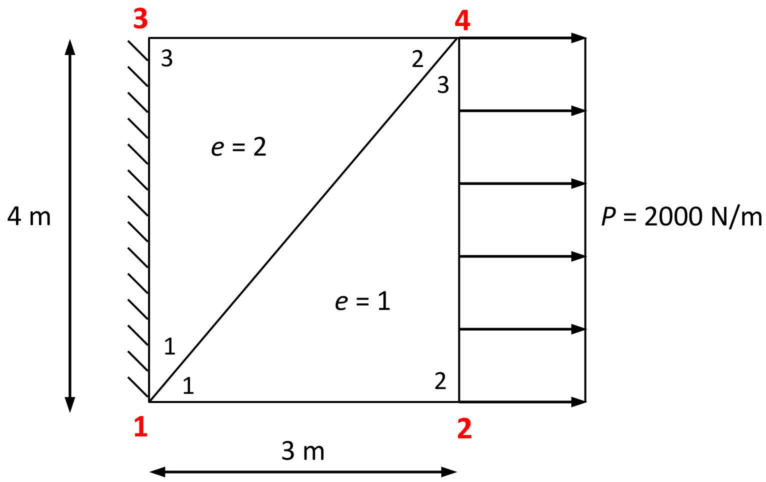
Due to the specified EBC, temperatures T_2 and T_4 are known to be 100. Let's reduce the system to a 2x2 system by getting rid of the known temperature values

$$\begin{bmatrix} 200 & -100 \\ -100 & 400 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 500 + 100 \times 100 \\ 9500 \end{Bmatrix}$$

Solving this system we get the following unknown temperatures

$$\begin{Bmatrix} T_1 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 73.57 \\ 42.14 \end{Bmatrix}$$

Example 2: Consider a thin elastic plate subjected to a uniformly distributed edge load. Consider a 2 element mesh as shown and calculate the nodal displacements. Thickness of the plate is 1 mm. Modulus of elasticity and the Poisson's ratio of the plate are 200 GPa and 0.25, respectively.



We first need to calculate the Jacobian matrix of each element. Let's take the origin of the coordinate system to be at the 1st global node of the FE mesh.

Jacobian calculation for the first element:

$$[J^1] = \begin{bmatrix} \frac{\partial S_1}{\partial \xi} & \frac{\partial S_2}{\partial \xi} & \frac{\partial S_3}{\partial \xi} \\ \frac{\partial S_1}{\partial \eta} & \frac{\partial S_2}{\partial \eta} & \frac{\partial S_3}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1^1 & y_1^1 \\ x_2^1 & y_2^1 \\ x_3^1 & y_3^1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 4 \end{bmatrix}$$

$$|J^1| = 12 \quad , \quad [J^1]^{-1} = \begin{bmatrix} 1/3 & 0 \\ -1/4 & 1/4 \end{bmatrix}$$

Jacobian calculation for the first element:

$$[J^2] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 4 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 4 \end{bmatrix}$$

$$|J^2| = 12 \quad , \quad [J^2]^{-1} = \begin{bmatrix} 1/3 & -1/3 \\ 0 & 1/4 \end{bmatrix}$$

Now we can calculate the elemental stiffness matrices. For this plane stress problem the $[c]$ matrix used to establish stress-strain relation is

$$[c] = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{6.4 \times 10^{11}}{3} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

Stiffness matrix calculation for the first element:

Shape functions for three-node triangular element are

$$S_1 = 1 - \xi - \eta \quad , \qquad S_2 = \xi \quad , \qquad S_3 = \eta$$

Stiffness matrix integrals contain the derivatives of the shape functions, $\frac{\partial S_i}{\partial x}$ and $\frac{\partial S_i}{\partial y}$.

$$\frac{\partial S_i}{\partial x} = \frac{\partial S_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial S_i}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial S_i}{\partial \xi}J_{11}^{-1} + \frac{\partial S_i}{\partial \eta}J_{12}^{-1} = \frac{1}{3}\frac{\partial S_i}{\partial \xi} + 0$$

which gives for different shape functions

$$\frac{\partial S_1^1}{\partial x} = \frac{1}{3}(-1) = -\frac{1}{3} \quad , \qquad \frac{\partial S_2^1}{\partial x} = \frac{1}{3}(1) = \frac{1}{3} \quad , \qquad \frac{\partial S_3^1}{\partial x} = \frac{1}{3}(0) = 0$$

Similarly derivatives with respect to *y* can be calculated

$$\frac{\partial S_i}{\partial y} = \frac{\partial S_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial S_i}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial S_i}{\partial \xi}J_{21}^{-1} + \frac{\partial S_i}{\partial \eta}J_{22}^{-1} = -\frac{1}{4}\frac{\partial S_i}{\partial \xi} + \frac{1}{4}\frac{\partial S_i}{\partial \eta}$$

$$\frac{\partial S_1^1}{\partial y} = -\frac{1}{4}(-1) + \frac{1}{4}(-1) = 0 \quad , \qquad \frac{\partial S_2^1}{\partial y} = -\frac{1}{4}(1) + \frac{1}{4}(0) = -\frac{1}{4} \quad ,$$
$$\frac{\partial S_3^1}{\partial y} = -\frac{1}{4}(0) + \frac{1}{4}(1) = \frac{1}{4}$$

Now we can use these derivatives in the stiffness matrix integrals.

$$K11_{ij}^1 = \int_{\Omega^e} \left(c_{11}\frac{\partial S_i}{\partial x}\frac{\partial S_j}{\partial x} + c_{33}\frac{\partial S_i}{\partial y}\frac{\partial S_j}{\partial y} \right) t^e \, dxdy$$

$$K11_{11}^1 = \int_{\Omega^e} \left(\left(\frac{6.4 \times 10^{11}}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) + \left(\frac{2.4 \times 10^{11}}{3}\right)(0)(0) \right)(10^{-3}) \, dxdy$$
$$= \int_{\Omega^e} \frac{6.4 \times 10^8}{27} \, dxdy = \left(\frac{6.4 \times 10^8}{27}\right)(\text{Area of element 1}) = 1.422 \times 10^8$$

Other entries of $K11^1$ matrix are calculated in a similar way to get

$$K11^1 = 10^8 \begin{bmatrix} 1.422 & -1.422 & 0 \\ & 1.722 & -0.3 \\ symm & & 0.3 \end{bmatrix}$$

Note that the above matrix is symmetric. We can use this property to save some computation time.

Now we can calculate $K12^1$

$$K12_{ij}^1 = \int_{\Omega^e} \left(c_{12}\frac{\partial S_i}{\partial x}\frac{\partial S_j}{\partial y} + c_{33}\frac{\partial S_i}{\partial y}\frac{\partial S_j}{\partial x} \right) t^e \, dxdy$$

$$K12^1 = 10^8 \begin{bmatrix} 0 & 0.267 & -0.267 \\ 0.4 & -0.667 & 0.267 \\ -0.4 & 0.4 & 0 \end{bmatrix}$$

As seen the above matrix is <u>NOT symmetric</u>.

We do not need to calculate $[K12]^1$ matrix because it is the transpose of $[K21]^1$

$$K21^1 = [K21^1]^T$$

Finally we calculate $[K22]^1$

$$K22^1_{ij} = \int_{\Omega^e} \left(c_{33} \frac{\partial S_i}{\partial x} \frac{\partial S_j}{\partial x} + c_{22} \frac{\partial S_i}{\partial y} \frac{\partial S_j}{\partial y} \right) t^e \, dxdy$$

$$K22^1 = 10^8 \begin{bmatrix} 0.533 & -0.533 & 0 \\ & 1.333 & -0.8 \\ symm & & 0.8 \end{bmatrix}$$

This one is a symmetric one.

When we bring these four sub-elemental stiffness matrices together we get the following stiffness matrix for the first element.

$$K^1 = \begin{bmatrix} 1.422 & -1.422 & 0 & 0 & 0.267 & -0.267 \\ & 1.722 & -0.3 & 0.4 & -0.667 & 0.267 \\ & & 0.3 & -0.4 & 0.4 & 0 \\ & & & 0.533 & -0.533 & 0 \\ & symm & & & 1.333 & -0.8 \\ & & & & & 0.8 \end{bmatrix}$$

<u>Stiffness matrix calculation for the second element:</u>

Derivatives of the shape functions for this element is calculated in a way similar to what we did for the first element.

$$\frac{\partial S_1^2}{\partial x} = 0 \quad , \quad \frac{\partial S_2^2}{\partial x} = \frac{1}{3} \quad , \quad \frac{\partial S_3^2}{\partial x} = -\frac{1}{3}$$

$$\frac{\partial S_1^2}{\partial y} = -\frac{1}{4} \quad , \quad \frac{\partial S_2^2}{\partial y} = 0 \quad , \quad \frac{\partial S_3^2}{\partial y} = \frac{1}{4}$$

Using these derivatives sub-elemental matrices are evaluated as

$$K11^2 = 10^8 \begin{bmatrix} 0.3 & 0 & -0.3 \\ & 1.422 & -1.422 \\ symm & & 1.722 \end{bmatrix}$$

$$K12^2 = 10^8 \begin{bmatrix} 0 & -0.4 & 0.4 \\ -0.267 & 0 & 0.267 \\ 0.267 & 0.4 & -0.667 \end{bmatrix}$$

$$K21^2 = [K12^2]^T$$

$$K22^2 = 10^8 \begin{bmatrix} 0.8 & 0 & -0.8 \\ & 0.533 & -0.533 \\ symm & & 1.333 \end{bmatrix}$$

Bringing these matrices together, elemental stiffness matrix for the second element becomes

$$K^2 = \begin{bmatrix} 0.3 & 0 & -0.3 & 0 & -0.4 & 0.4 \\ & 1.422 & -1.422 & -0.267 & 0 & 0.267 \\ & & 1.722 & 0.267 & 0.4 & -0.667 \\ & & & 0.8 & 0 & -0.8 \\ & symm & & & 0.533 & -0.533 \\ & & & & & 1.333 \end{bmatrix}$$

We don't need to calculate any force vectors because we assume that no body forces act on the plate.

Elemental boundary integral vector needs to be calculated only for the first element due to the specified distributed force. The given traction is uniform over one face of the element and we know how to calculate boundary integrals of such secondary variables from the previous solved example. Considering the following

$$Q_2^1|_{face\ 2} = \frac{(P)(L_{face\ 2})}{2} = \frac{(2000)(4)}{2} = 4000\ N$$

This is actually the point force at the bottom right corner of the plate if the distributed force is distributed to the nodes of the face it applies.

The other half of the distributed should apply to the second node of the face

$$Q_3^1|_{face\ 2} = \frac{(P)(L_{face\ 2})}{2} = 4000\ N$$

No other boundary should be calculated.

Now we can assemble the two elemental systems to get the following 8x8 global system usıng the following local to global unknown mapping matrix

$$LtoGlong = \begin{bmatrix} 1 & 2 & 4 & 5 & 6 & 8 \\ 1 & 4 & 3 & 5 & 8 & 7 \end{bmatrix}$$

Global system is

$$10^8 \begin{bmatrix} 1.422 + 0.3 & -1.422 & -0.3 & 0+0 & 0+0 & 0.267 & 0.4 & -0.267 - 0.4 \\ & 1.722 & 0 & -0.3 & 0.4 & -0.667 & 0 & 0.267 \\ & & 1.722 & -1.422 & 0.267 & 0 & -0.667 & 0.4 \\ & & & 0.3 + 1.422 & -0.4 - 0.267 & 0.4 & 0.267 & 0+0 \\ & & & & 0.533 + 0.8 & -0.533 & 0 - 0.8 & 0 \\ & symm & & & & 1.333 & 0 & -0.8 \\ & & & & & & 1.333 & -0.533 \\ & & & & & & & 0.8 + 0.533 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ 4000 \\ Q_3 \\ 4000 \\ Q_5 \\ 0 \\ Q_7 \\ 0 \end{Bmatrix}$$

Q_1 and Q_5 are the horizontal and vertical reaction forces at node 1 and they are unknown. Q_3 and Q_7 are the horizontal and vertical reaction forces at node 3 and they are also unknown. On the other hand we have the following EBCs

$$U_1 = U_3 = U_5 = U_7 = 0$$

Therefore we only need to solve for U_2, U_4, U_6 and U_8. The 8x8 global system can be reduced to the following 4x4 system due to the EBCs

$$10^8 \begin{bmatrix} 1.722 & -0.3 & -0.667 & 0.267 \\ & 1.722 & 0.4 & 0 \\ & & 1.333 & -0.8 \\ symm & & & 1.333 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_4 \\ U_6 \\ U_8 \end{Bmatrix} = \begin{Bmatrix} 4000 \\ 4000 \\ 0 \\ 0 \end{Bmatrix}$$

Solving the above system we find the unknown displacements as

$$\begin{Bmatrix} U_2 \\ U_4 \\ U_6 \\ U_8 \end{Bmatrix} = 10^{-4} \begin{Bmatrix} 0.3050 \\ 0.2731 \\ 0.0531 \\ -0.0292 \end{Bmatrix} m$$

Due to the symmetry of the plate geometry and loading, displacements should also be symmetric with respect to $y = 2\ m$ line. However, the approximate solution does not show this symmetry property. This is due to the coarse and asymmetric FE mesh. A symmetric mesh should provide symmetric results.

Since we used CST elements, strains should be constant over each element. For example for the first element

$$\varepsilon_{xx}^1 = \frac{\partial u}{\partial x} = \sum_{j=1}^{3} \frac{\partial S_j^1}{\partial x} u_j^1 = \frac{\partial S_1^1}{\partial x} U_1 + \frac{\partial S_1^1}{\partial x} U_2 + \frac{\partial S_3^1}{\partial x} U_4$$

$$= \left(-\frac{1}{3}\right)(0) + \left(\frac{1}{3}\right)(0.3050 \times 10^{-4}) + (0)(0.2731 \times 10^{-4}) = 0.1017 \times 10^{-4}$$

$$\varepsilon_{yy}^1 = \frac{\partial v}{\partial y} = \sum_{j=1}^{3} \frac{\partial S_j^1}{\partial y} v_j^1 = \frac{\partial S_1^1}{\partial y} U_5 + \frac{\partial S_1^1}{\partial y} U_6 + \frac{\partial S_3^1}{\partial y} U_8$$

$$= (0)(0) + \left(-\frac{1}{4}\right)(0.0531 \times 10^{-4}) + \left(\frac{1}{4}\right)(-0.0292 \times 10^{-4}) = 0.0206 \times 10^{-4}$$

Angular strain γ_{xy} can be calculated in a similar way.

After calculating strains, stresses can now be calculated. They will also be constant over each element. For the first element

$$\sigma_{xx}^1 = c_{11}\varepsilon_{xx}^1 + c_{12}\varepsilon_{yy}^1$$

$$= \left(\frac{6.4\times10^{11}}{3}\right)(0.1017\times10^{-4}) + \left(\frac{6.4\times10^{11}}{3}0.25\right)(0.0206\times10^{-4}) = 2.28\ MPa$$

$$\sigma_{yy}^1 = c_{12}\varepsilon_{xx}^1 + c_{22}\varepsilon_{yy}^1$$

$$= \left(\frac{6.4\times10^{11}}{3}0.25\right)(0.1017\times10^{-4}) + \left(\frac{6.4\times10^{11}}{3}\right)(0.0206\times10^{-4}) = 0.98\ MPa$$

Shear stress τ_{xy}^1 can be calculated in a similar way. Finally principle stresses or von Mises stresses over each element can be calculated and checked against the yield strength of the plate for a possible failure.