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TRANSIENT TWO-DIMENSIONAL HEAT CONDUCTION PROBLEMS SOLVED BY THE FINITE ELEMENT METHOD

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SUMMARY

A finite element weighted residual process has been used to solve transient linear and non-linear two-dimensional heat conduction problems. Rectangular prisms in a space-time domain were used as the finite elements. The weighting function was equal to the shape function defining the dependent variable approximation. The results are compared in tables with analytical, as well as other numerical data. The finite element method compared favourably with these results. It was found to be stable, convergent to the exact solution, easily programmed, and computationally fast. Finally, the method does not require constant parameters over the entire solution domain.

INTRODUCTION

In many instances simplifications which would reduce physical heat conduction problems to problems depending on only one space co-ordinate and time may not be possible, and transient conduction in more than one space dimension must be considered. For example, this would occur when end or edge effects are significant so as to affect the desired results.

Although a great number of these two-dimensional problems have been solved analytically, only a limited number of geometrical shapes and only those boundary conditions which can be easily expressed mathematically can be handled. There are many transient heat conduction problems of considerable practical value for which no analytical solution is feasible, and resort is made to numerical and analogue techniques. This paper will consider the former only.

Several different techniques of numerical analysis of transient heat conduction problems in two-dimensions exist. To date the finite difference and finite element techniques have been the most prominent. For the finite difference method[1] these include, to name a few, the explicit method, the implicit method, and the implicit alternating-direction method. The finite element method was first applied by Wilson and Nickell.[2] Their method for analysing the unsteady flow of heat was based on a variational principle by Gurtin.[3] Richardson and Shum[4] used the same variational principle and the finite element method to solve transient heat conduction problems involving non-linear boundary conditions. Emery and Carson[5] and Visser[6] used variational formulations in their finite element solutions to non-stationary temperature distribution problems. Other examples of the finite element method applied to transient heat conduction problems are referenced in Zienkiewicz[7] and Desai and Abel.[8]

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This paper discusses a simple finite element technique for solving two-dimensional linear and non-linear transient heat flow problems. The method is a finite element weighted residual process using finite elements in a space-time domain. Bruch and Zyvoloski[9] used this technique on one-dimensional linear field problems, one of which was a heat flow problem. They give a comparison of their finite element results with exact solutions and finite difference results.

FINITE ELEMENT FORMULATION

The two-dimensional heat conduction equation which describes the unsteady temperature distribution in a solid in domain R (Figure 1) is governed by the following differential equation

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+Q=\rho c\frac{\partial T}{\partial t} \tag{1}$$

and subject to the conditions on the boundary surface S,

$$T=T_S \text{ on } S_1 \tag{1a}$$

$$-k_x\frac{\partial T}{\partial x}l_x-k_y\frac{\partial T}{\partial y}l_y=q \quad \text{on} \quad S_2 \tag{1b}$$

$$-k_x\frac{\partial T}{\partial x}l_x-k_y\frac{\partial T}{\partial y}l_y=\alpha(T-T_a) \quad \text{on} \quad S_3 \tag{1c}$$

and the initial condition

$$T(x,y,0)=T_0(x,y) \tag{1d}$$

where $S=S_1+S_2+S_3$; S_1 is the part of the boundary on which T is prescribed; S_2 is the part of the boundary on which q, the intensity of heat input, is prescribed; S_3 is the part of the boundary on which $\alpha(T-T_a)$ is prescribed; $T(x,y,t)$ is the temperature in the solid; k_x and k_y are specified thermal conductivities where x and y are the principal directions of the conductivity tensor; c is the specific heat; ρ is the density; Q is the externally applied heat flux; T_S is a given boundary surface temperature; T_0 is the initial temperature; l_x and l_y are the direction cosines of the outward normal to the boundary surface; α is the heat transfer coefficient; and T_a is the temperature of the surroundings.

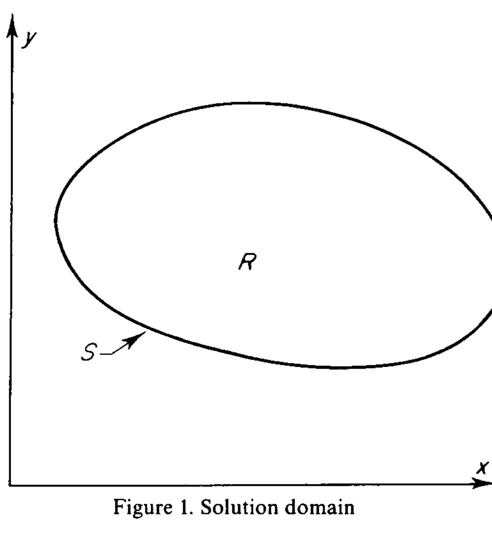

Figure 1. Solution domain

The approach used to solve equation (1) subject to the auxiliary conditions, equations (1a), (1b), (1c), and (1d) is similar to that suggested by Zienkiewicz and Parekh.[10] A finite element representation based on the Galerkin principle is obtained without recourse to variational theorems. Dividing the solution domain into finite elements in space and time (Figure 2), the temperature is approximated within each element by

$$T(x, y, t) = \sum_{I}^{m} N_I(x, y, t)T_I \tag{2}$$

where N_I are the usual shape functions defined piecewise, element by element; I is a summation subscript; T_I is the discrete nodal representation of $T(x, y, t)$; and m is the number of nodes in an element.

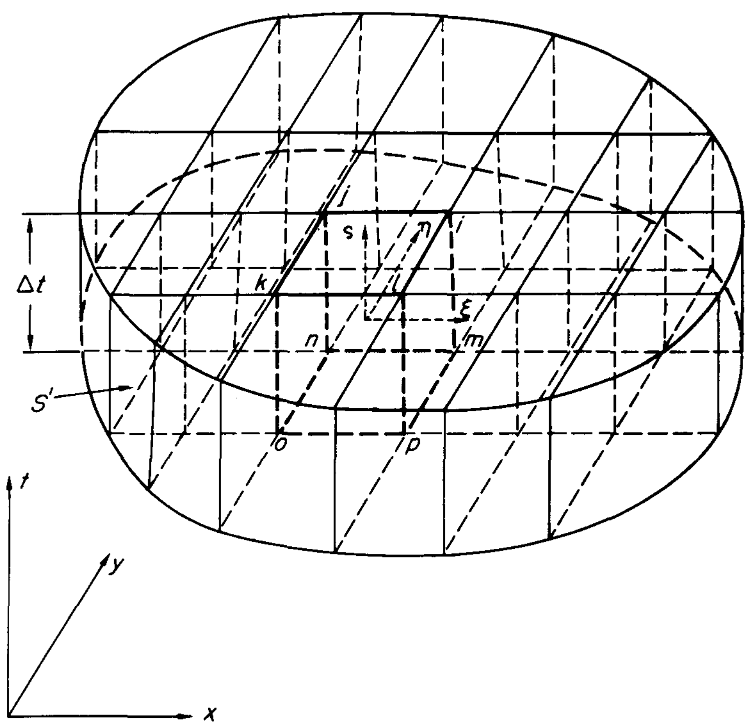

Figure 2. Solution domain divided into rectangular prismatic elements in space and time

Using the weighted residual process in which the weighting function is equal to the shape function defining the approximation, the Galerkin representation for the heat problem (equation (1)) is

$$\int_{R'} N_I\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+Q-\rho c\frac{\partial T}{\partial y}\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0 \tag{3}$$

where R' is the finite element solution domain, and $\mathrm{d}R' = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$. Using integration by parts on the first and second terms in equation (3), the equation simplifies to

$$-\int_{R'}\left[k_x\frac{\partial N_I}{\partial x}\frac{\partial T}{\partial x}+k_y\frac{\partial N_I}{\partial y}\frac{\partial T}{\partial y}-N_IQ+N_I\rho c\frac{\partial T}{\partial t}\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$-\int_{S'_2} N_I q \, dS'_2 - \int_{S'_3} N_I \alpha (T - T_a) \, dS'_3 = 0 \tag{4}$$

where S'_2 and S'_3 are segments of the external surface area S' of the finite element solution region (see Figure 2). Inserting the temperature approximation, the Ith equation of the simultaneous equations that will allow the solution for the n values of T_I is

$$-\int_{R'} \left[k_x \frac{\partial N_I}{\partial x} \frac{\partial}{\partial x}\left(\sum_1^n N_I T_I\right) + k_y \frac{\partial N_I}{\partial y} \frac{\partial}{\partial y}\left(\sum_1^n N_I T_I\right) - N_I Q + N_I \rho c \frac{\partial}{\partial t}\left(\sum_1^n N_I T_I\right)\right]$$
$$\times \, dx \, dy \, dt - \int_{S'_2} N_I q \, dS'_2 - \int_{S'_3} N_I \left(\sum_1^n N_I T_I - T_a\right) dS'_3 = 0 \tag{5}$$

The resulting system will consist of n linear algebraic equations in n unknown.

NUMERICAL SOLUTIONS

The technique to solve the problem described by equation (5) in the finite element network shown in Figure 2 is as follows.[9,11] Since values of temperature are known at t_0, values at $t_1 = t_0 + \Delta t$ can be obtained by summing around the nodes at this latter step and then solving the system of simultaneous linear algebraic equations that results. At each new time step, an identical procedure is used until a required time is reached.

This technique will be applied to three examples whose solution domain will be square (this, however, is not a restriction on the method) and where Q and α are equal to zero with $c = 1{\cdot}0$ Btu/(m^3 °F).

The temperature approximation using rectangular prisms as the elements in the finite element method and linear shape functions within an element is

$$\begin{aligned} T(x, y, t) = {} & \tfrac{1}{8}(1+\xi)(1+\eta)(1+\zeta)T_i + \tfrac{1}{8}(1-\xi)(1+\eta)(1+\zeta)T_j + \tfrac{1}{8}(1-\xi)(1-\eta)(1+\zeta)T_k \\ & + \tfrac{1}{8}(1+\xi)(1-\eta)(1+\zeta)T_l + \tfrac{1}{8}(1+\xi)(1+\eta)(1-\zeta)T_m + \tfrac{1}{8}(1-\xi)(1+\eta)(1-\zeta)T_n \\ & + \tfrac{1}{8}(1-\xi)(1-\eta)(1-\zeta)T_0 + \tfrac{1}{8}(1+\xi)(1-\eta)(1-\zeta)T_p = N_i T_i + N_j T_j \\ & + N_k T_k + N_l T_l + N_m T_m + N_n T_n + N_o T_o + N_p T_p \end{aligned}$$

where $\xi = 2(x - x_c)/(\Delta x)$; $\eta = 2(y - y_c)/(\Delta y)$; $\zeta = 2(t - t_c)/(\Delta t)$; x_c, y_c, t_c = co-ordinates of the centroid of each element; $T_i, T_j, T_k, T_l, T_m, T_n, T_o, T_p$ = values of temperatures at the appropriate nodal points (Figure 2); $N_i, N_j, N_k, N_l, N_m, N_n, N_o, N_p$ = shape functions; Δx = x-co-ordinate spacing; Δy = y-co-ordinate spacing; Δt = t-co-ordinate spacing; and i, j, k, l, m, n, o, p = node numbers.

The first two-dimensional heat conduction problem to be solved is one governed by equation (1) and subject to the following boundary and initial conditions

$$T(0, y, t) = T(x, 0, t) = T(L_x, y, t) = T(x, L_y, t) = 0 \tag{6a}$$

and

$$T(x, y, 0) = 30 \tag{6b}$$

where L_x and L_y are the lengths of the solution domain in the x and y directions, respectively. The analytical solution for this problem is

$$T(x, y, t) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} A_n \sin \frac{n\pi x}{L_x} \sin \frac{j\pi y}{L_y} \exp\left[-\left(\frac{k_x n^2 \pi^2}{L_x^2} + \frac{k_y j^2 \pi^2}{L_y^2}\right)t\right] \tag{7}$$

where

$$A_n = \frac{4(30)}{nj\pi^2}[(-1)^n - 1][(-1)^j - 1] \tag{7a}$$

Table I lists the analytical results using equation (7) for the square domain $0 \leqslant x \leqslant 3{\cdot}0$ m, $0 \leqslant y \leqslant 3{\cdot}0$ m, with $k_x = k_y = 1{\cdot}25$ Btu/(hr m °F), $\Delta x = \Delta y = 0{\cdot}3$ m, and $t = 1{\cdot}2$ hr. Tables II and III list the finite element weighted residual solution for the same problem using two different time steps $\Delta t = 0{\cdot}1$ hr and $\Delta t = 0{\cdot}05$ hr, respectively.

Table I. Temperatures at $t = 1{\cdot}2$ hr obtained using the analytical solution equations (7)

0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
0·000	0·173	0·329	0·453	0·533	0·560	0·533	0·453	0·329	0·173	0·000
0·000	0·329	0·626	0·862	1·013	1·065	1·013	0·862	0·626	0·329	0·000
0·000	0·453	0·862	1·186	1·394	1·466	1·394	1·186	0·862	0·453	0·000
0·000	0·533	1·013	1·394	1·639	1·723	1·639	1·394	1·013	0·553	0·000
0·000	0·560	1·065	1·466	1·723	1·812	1·723	1·466	1·065	0·560	0·000
0·000	0·533	1·013	1·394	1·639	1·723	1·639	1·394	1·013	0·533	0·000
0·000	0·453	0·862	1·186	1·394	1·466	1·394	1·186	0·862	0·453	0·000
0·000	0·329	0·626	0·862	1·013	1·065	1·013	0·862	0·626	0·329	0·000
0·000	0·173	0·329	0·453	0·533	0·560	0·533	0·453	0·329	0·173	0·000
0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000

Table II. Temperatures at $t = 1{\cdot}2$ hr obtained using the finite element technique with $\Delta t = 0{\cdot}1$ hr

0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
0·000	0·201	0·383	0·527	0·620	0·652	0·620	0·527	0·383	0·201	0·000
0·000	0·383	0·728	1·003	1·179	1·239	1·179	1·003	0·728	0·383	0·000
0·000	0·527	1·003	1·380	1·622	1·706	1·622	1·380	1·003	0·527	0·000
0·000	0·620	1·179	1·622	1·907	2·005	1·907	1·622	1·179	0·620	0·000
0·000	0·652	1·239	1·706	2·005	2·108	2·005	1·706	1·239	0·652	0·000
0·000	0·620	1·179	1·622	1·907	2·005	1·907	1·622	1·179	0·620	0·000
0·000	0·527	1·003	1·380	1·622	1·706	1·622	1·380	1·003	0·527	0·000
0·000	0·383	0·728	1·003	1·179	1·239	1·179	1·003	0·728	0·383	0·000
0·000	0·201	0·383	0·527	0·620	0·652	0·620	0·527	0·383	0·201	0·000
0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000

Table III. Temperatures at $t = 1{\cdot}2$ hr obtained using the finite element technique with $\Delta t = 0{\cdot}05$ hr

0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
0·000	0·185	0·352	0·485	0·570	0·599	0·570	0·485	0·352	0·185	0·000
0·000	0·352	0·670	0·922	1·083	1·139	1·083	0·922	0·670	0·352	0·000
0·000	0·485	0·922	1·268	1·491	1·568	1·491	1·269	0·922	0·485	0·000
0·000	0·570	1·083	1·491	1·753	1·843	1·753	1·491	1·083	0·570	0·000
0·000	0·599	1·139	1·568	1·843	1·938	1·843	1·568	1·139	0·599	0·000
0·000	0·570	1·083	1·491	1·753	1·843	1·753	1·491	1·083	0·570	0·000
0·000	0·485	0·922	1·268	1·491	1·568	1·491	1·269	0·922	0·485	0·000
0·000	0·352	0·670	0·922	1·083	1·139	1·083	0·922	0·670	0·352	0·000
0·000	0·185	0·352	0·485	0·570	0·599	0·570	0·485	0·352	0·185	0·000
0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000

A second example will be one governed by equation (1) and subject to the initial condition, equation (6b), and the following boundary conditions

$$T(x, 0, t) = T(L_x, y, t) = T(x, L_y, t) = 0 \tag{8a}$$

and

$$\frac{\partial T}{\partial x}(0, y, t) = 0 \tag{8b}$$

The analytical solution for the problem is

$$T(x, y, t) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} B_n \cos \frac{(2n-1)\pi x}{2L_x} \sin \frac{j\pi y}{L_y} \exp\left[-\left(\frac{k_x(2n-1)^2\pi^2}{4L_x^2} + \frac{k_y j^2\pi^2}{L_y^2}\right)t\right] \tag{9}$$

where

$$B_n = \frac{8(30)}{\pi^2 j(2n-1)}(-1)^{n+2}[(-1)^j - 1] \tag{9a}$$

Table IV lists the analytical results using equations (9) for the square domain $0 \leqslant x \leqslant 3{\cdot}0$ m, $0 \leqslant y \leqslant 3{\cdot}0$ m, with $k_x = k_y = 1{\cdot}25$ Btu/(hr m °F), $\Delta x = \Delta y = 0{\cdot}3$ m, and $t = 1{\cdot}2$ hr. Tables V and VI list the finite element weighted residual solution for the same problem using two different time steps $\Delta t = 0{\cdot}1$ hr and $\Delta t = 0{\cdot}05$ hr, respectively.

Table IV. Temperatures at $t = 1{\cdot}2$ hr obtained using the analytical solution equations (9)

0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
1·899	1·878	1·815	1·710	1·563	1·377	1·153	0·897	0·614	0·312	0·000
3·612	3·572	3·452	3·252	2·973	2·618	2·193	1·705	1·167	0·593	0·000
4·972	4·916	4·751	4·476	4·092	3·604	3·018	2·347	1·606	0·816	0·000
5·844	5·780	5·585	5·261	4·810	4·237	3·548	2·759	1·888	0·959	0·000
6·145	6·077	5·872	5·532	5·058	4·455	3·731	2.901	1·985	1·009	0·000
5·844	5·780	5·585	5·261	4·810	4·237	3·548	2·759	1·888	0·959	0·000
4·972	4·916	4·751	4·476	4·092	3·604	3·018	2·347	1·606	0·816	0·000
3·612	3·572	3·452	3·252	2·973	2·618	2·193	1·705	1·167	0·593	0·000
1·899	1·878	1·815	1·710	1·563	1·377	1·153	0·897	0·614	0·312	0·000
0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000

Table V. Temperatures at $t = 1{\cdot}2$ hr obtained using the finite element technique with $\Delta t = 0{\cdot}1$ hr

0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
2·028	2·006	1·940	1·830	1·677	1·480	1·242	0·967	0·663	0·337	0·000
3·857	3·815	3·690	3·482	3·189	2·815	2·362	1·840	1·261	0·641	0·000
5·308	5·251	5·079	4·792	4·390	3·874	3·251	2·533	1·736	0·882	0·000
6·240	6·173	5·971	5·633	5·160	4·554	3·822	2·977	2·040	1·037	0·000
6·561	6·491	6·278	5·923	5·426	4·788	4·019	3·131	2·145	1·091	0·000
6·240	6·173	5·971	5·633	5·160	4·554	3·822	2·977	2·040	1·037	0·000
5·308	5·251	5·079	4·792	4·390	3·874	3·251	2·533	1·736	0·882	0·000
3·857	3·815	3·690	3·482	3·189	2·815	2·362	1·840	1·261	0·641	0·000
2·028	2·006	1·940	1·830	1·677	1·480	1·242	0·967	0·663	0·337	0·000
0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000

Table VI. Temperatures at $t = 1{\cdot}2$ hr obtained using the finite element technique with $\Delta t = 0{\cdot}05$ hr

0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
1·951	1·929	1·865	1·758	1·609	1·418	1·189	0·925	0·633	0·322	0·000
3·711	3·670	3·548	3·344	3·060	2·697	2·261	1·760	1·205	0·612	0·000
5·107	5·051	4·883	4·603	4·212	3·713	3·112	2·422	1·658	0·843	0·000
6·004	5·938	5·741	5·411	4·951	4·364	3·659	2·847	1·950	0·991	0·000
6·313	6·244	6·036	5·690	5·206	4·589	3·847	2·994	2·050	1·042	0·000
6·004	5·938	5·741	5·411	4·951	4·364	3·659	2·847	1·950	0·991	0·000
5·107	5·051	4·883	4·603	4·212	3·713	3·112	2·422	1·658	0·843	0·000
3·711	3·670	3·548	3·344	3·060	2·697	2·261	1·760	1·205	0·612	0·000
1·951	1·929	1·865	1·758	1·609	1·418	1·189	0·925	0·633	0·322	0·000
0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000

The last example will be one governed by equation (1) and subject to the following boundary and initial conditions

$$T(L_x, y, t) = T(x, L_y, t) = 1{\cdot}0 \tag{10a}$$

$$\frac{\partial T(0, y, t)}{\partial x} = \frac{\partial T(x, 0, t)}{\partial y} = 0{\cdot}0 \tag{10b}$$

and

$$T(x, y, 0) = 0{\cdot}0 \tag{10c}$$

This example describes the unsteady heat conduction in a long bar of square cross-section. In order to check these results with those of Carnahan *et al*,[1] who used an implicit alternating direction finite difference method, it is assumed that all the constants are 1·0 and the dependent and independent variables are dimensionless in equations (1), (10), and (11). The analytical solution for this problem is

$$T(x, y, t) = 1{\cdot}0 + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} C_n \cos\frac{(2n-1)\pi x}{2L_x} \cos\frac{(2j-1)\pi y}{2L_y} \exp\left[-\left(\frac{k_x(2n-1)^2\pi^2}{4L_x^2} + \frac{k_y(2j-1)^2\pi^2}{4L_y^2}\right)t\right] \tag{11}$$

where

$$C_n = \frac{16{\cdot}0(-1{\cdot}0)(-1)^{n+1}(-1)^{j+1}}{\pi^2(2n-1)(2j-1)} \tag{11a}$$

Table VII lists the analytical results using equations (11) for the square domain $0 \leqslant x \leqslant 1{\cdot}0$, $0 \leqslant y \leqslant 1{\cdot}0$, with $k_x = k_y = 1{\cdot}0$, $\Delta x = \Delta y = 0{\cdot}1$, and $t = 0{\cdot}75$. Table VIII lists the finite element weighted residual solution for the same problem using a time step $\Delta t = 0{\cdot}05$. Table IX lists the results of Carnahan *et al*[1] who used an implicit alternating direction finite difference method with the same time step.

The finite element scheme used in the above examples is an implicit scheme and is stable. For the element size used, the results check closely with the analytical results, e.g. in the second example using a $\Delta t = 0{\cdot}05$ hr, the maximum deviation from the analytical results was about 2·73 per cent. By decreasing the element size, the finite element solution will converge to the exact solution. This can be seen from comparing Tables II and III with Table I, and Tables V and VI with Table IV in which only the time step size was decreased.

Table VII. Dimensionless temperatures at $t = 0{\cdot}75$ obtained using the analytical solution Equations (11)

0·960	0·960	0·962	0·964	0·968	0·972	0·976	0·982	0·988	0·994	1·000
0·960	0·961	0·962	0·965	0·968	0·972	0·977	0·982	0·988	0·994	1·000
0·962	0·962	0·964	0·966	0·969	0·973	0·978	0·983	0·988	0·994	1·000
0·964	0·965	0·966	0·968	0·971	0·975	0·979	0·984	0·989	0·994	1·000
0·968	0·968	0·969	0·971	0·974	0·977	0·981	0·985	0·990	0·995	1·000
0·972	0·972	0·973	0·975	0·977	0·980	0·983	0·987	0·991	0·996	1·000
0·976	0·977	0·978	0·979	0·981	0·983	0·986	0·989	0·993	0·996	1·000
0·982	0·982	0·983	0·983	0·985	0·987	0·989	0·992	0·994	0·997	1·000
0·988	0·988	0·988	0·989	0·990	0·991	0·993	0·994	0·996	0·998	1·000
0·994	0·994	0·994	0·994	0·995	0·996	0·996	0·997	0·998	0·999	1·000
1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000

Table VIII. Dimensionless temperatures at $t = 0{\cdot}75$ obtained using the finite element technique with $\Delta t = 0{\cdot}05$

0·953	0·953	0·955	0·958	0·962	0·967	0·972	0·979	0·985	0·993	1·000
0·953	0·954	0·956	0·958	0·962	0·967	0·973	0·979	0·986	0·993	1·000
0·955	0·956	0·957	0·960	0·964	0·968	0·974	0·980	0·986	0·993	1·000
0·958	0·958	0·960	0·962	0·966	0·970	0·975	0·981	0·987	0·993	1·000
0·962	0·962	0·964	0·966	0·969	0·973	0·977	0·983	0·988	0·994	1·000
0·967	0·967	0·968	0·970	0·973	0·976	0·980	0·985	0·990	0·995	1·000
0·972	0·973	0·974	0·975	0·977	0·980	0·984	0·987	0·991	0·996	1·000
0·979	0·979	0·980	0·981	0·983	0·985	0·987	0·990	0·993	0·997	1·000
0·985	0·986	0·986	0·986	0·988	0·990	0·991	0·993	0·995	0·998	1·000
0·993	0·993	0·993	0·993	0·994	0·995	0·996	0·997	0·998	0·999	1·000
1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000

Table IX. Dimensionless temperatures at $t = 0{\cdot}75$ obtained by Carnahan *et al*[1] using an implicit alternating direction finite difference method with $\Delta t = 0{\cdot}05$

0·960	0·961	0·962	0·964	0·968	0·972	0·976	0·982	0·987	0·994	1·000
0·961	0·961	0·962	0·965	0·968	0·972	0·977	0·982	0·987	0·994	1·000
0·962	0·962	0·964	0·966	0·969	0·973	0·978	0·983	0·988	0·995	1·000
0·964	0·965	0·966	0·968	0·971	0·975	0·979	0·984	0·988	0·995	1·000
0·968	0·968	0·969	0·971	0·974	0·977	0·981	0·986	0·990	0·995	1·000
0·972	0·972	0·973	0·975	0·977	0·980	0·983	0·987	0·991	0·996	1·000
0·976	0·977	0·978	0·979	0·981	0·983	0·986	0·990	0·992	0·997	1·000
0·982	0·982	0·983	0·984	0·986	0·987	0·990	0·992	0·994	0·997	1·000
0·987	0·987	0·988	0·988	0·990	0·991	0·992	0·994	0·996	0·998	1·000
0·994	0·994	0·995	0·995	0·995	0·996	0·997	0·997	0·998	0·999	1·000
1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000

As is evident from Tables VIII and IX, both the finite element technique and the implicit alternating direction finite difference method compare within a fraction of a per cent with the analytical results.

Since both the implicit alternating direction finite difference method and the finite element method used on example three are stable methods, they exhibited, for the rather large time step used, the phenomenon of giving impossible temperatures, greater than 1, just inside the surface of the bar at the end of the first time step.[1] An oscillation of decreasing amplitude in the temperature took place for a number of time steps and then disappeared. The amplitudes of the temperatures in the oscillations for the finite element case were less and disappeared sooner than for the finite difference method.

As another example of the application of the technique, the following problem with sharp corners in the boundary is investigated, see Figure 3. Consider finding $T(x, y, t)$ satisfying the equation

$$\nabla^2 T = \frac{\partial T}{\partial t} \tag{12}$$

in the domain R, shown in Figure 3, with the boundary conditions

$$T(0, y, t) = 1{,}000 \tag{12a}$$

$$T(1, y, t) = 0 \tag{12b}$$

and $\partial T/\partial n = 0$ on all other boundaries, where $\partial/\partial n$ is the derivative normal to the boundary. The initial condition is a small time solution in a plane medium and is taken to be

$$T(x, y, 0) = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) \tag{12c}$$

where $t = 0{\cdot}0005$ and is equivalent to one time step in the numerical solutions that follow.

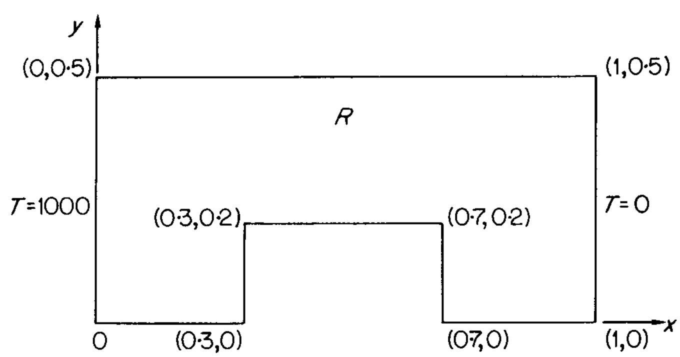

Figure 3. Solution domain

G. E. Bell[12] presents a method for treating the singularities which occur in the solution to this problem due to the sharp corners in the boundary. His method is essentially an extension of the method due to Motz[13] for solving elliptic problems and approximates to the analytical form of the singularity in terms of neighbouring function values at each time step. His method is used in conjunction with the simple explicit finite-difference scheme and subsequently the overall method is explicit.

Bell's[12] results (using 5 term approximation, 6 term approximation, and simple explicit schemes) are given in Table X along with results obtained herein using the finite element space-time co-ordinate method for the x and y spacings and time step shown.

A second example solved by Bell[12] is that shown in Figure 4. Here the singularities are closer together and this severely restricts the number of terms that can be used in the approximation.

Table X. Solution corresponding to a time of 0·1 sec (200 time steps)

	842	842	688	687	540	539	404	403	287	286	192	192	120	120	68	68	30	30	
1000																			0·000
	842	842	688	686	540	538	403	403	286	285	192	191	120	119	68	67	30	30	
	846	845	694	692	546	544	408	406	288	287	192	191	119	119	66	66	29	29	
1000																			0·000
	846	845	694	692	545	544	407	406	287	287	191	190	118	117	66	65	29	29	
	856	855	714	711	568	565	418	416	291	290	190	190	112	112	60	61	26	27	
1000																			0·000
	857	856	715	713	566	566	416	417	290	289	190	188	112	111	59	59	26	25	
	874	871	756	749	635	622	427	423	293	292	189	187	88	91	46	48	21	21	
1000																			0·000
	875	873	758	755	635	634	424	425	291	291	188	186	86	86	45	45	20	20	
	891	888	801	795	762	752							38	41	30	31	15	16	
1000																			0·000
	892	890	805	800	767	762							36	36	28	29	14	15	
	898	895	818	813	787	780							29	31	24	25	13	13	
1000																			0·000
	899	897	822	817	791	787							27	28	23	23	12	13	

5 term approx.	Simple Explicit
6 term approx.	Finite element (space-time co-ordinates)

$\Delta x = \Delta y = 0{\cdot}05$
$\Delta t = 0{\cdot}0005$

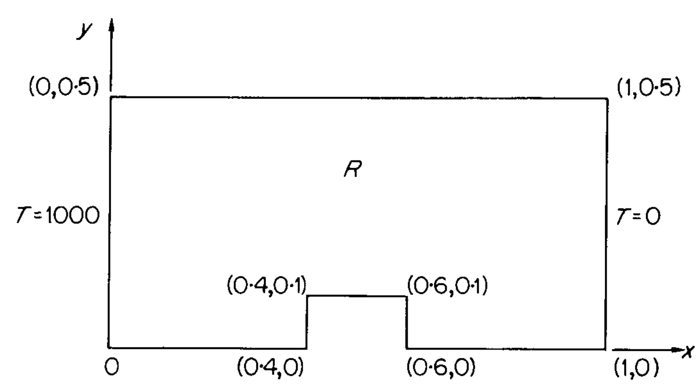

Figure 4. Solution domain

However, using a point configuration similar to the one he used on the previous example, he obtained the results shown in Table XI. These results are approximately equal to those he obtained using a 6 term approximation and a mesh, $\Delta x = \Delta y = 0{\cdot}025$.

Also tabulated in Table XI are results obtained by Bell[12] using the standard explicit method, by Wilson and Nickell[2] using a finite element scheme in which triangular elements of side 0·1 with linear shape functions were used, and by the finite element techniques given herein.

If $k_x = \psi_1 T$ and $k_y = \psi_2 T$ are inserted into equation (1), it becomes the non-linear partial differential equation

$$\frac{\partial}{\partial x}\left(\psi_1 T\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\psi_2 T\frac{\partial T}{\partial y}\right)+Q = \rho c\frac{\partial T}{\partial t} \tag{13}$$

where ψ_1 and ψ_2 are assumed to be constants. Take $\psi_1 = \psi_2 = 1{\cdot}0$ (Btu/hr m °F^2), $Q = 0$, $\rho c = 1{\cdot}0$ (Btu/m^2 °F), and the following auxiliary conditions

$$T(0, y, t) = T(x, 0, t) = T(L_x, y, t) = T(x, L_y, t) = 1{\cdot}0 \tag{13a}$$

and

$$T(x, y, 0) = 0{\cdot}1 \tag{13b}$$

The system of non-linear algebraic equations that results from the application of the finite element scheme was solved using a computer program given by Powell.[14] The results are presented in Tables XII and XIII for $t = 0{\cdot}2$ hr and $t = 0{\cdot}4$ hr, respectively.

Table XI. Solution corresponding to a time of 0·1 sec with $\Delta t = 0{\cdot}0005$

535	533	468	466	402	400	337	336	277	276	223	223	176	175	137	136	106	105
536	534			408	400			282	277			178	176			110	105
546	544	482	481	414	413	343	345	279	278	221	218	170	168	129	128	100	99
560	557	506	503	449	447	347	347	279	278	219	217	148	147	114	114	92	92
555	556			446	446			285	279			156	147			100	92
574	568	533	524	515	509							105	107	98	101	84	85
578	572	541	534	527	521							98	99	93	94	81	83
571	565			505	522							118	99			90	87

6 term approx. $\Delta x = \Delta y = 0{\cdot}05$	Finite element (Space-time co-ordinates) $\Delta x = \Delta y = 0{\cdot}05$
Finite element (Wilson & Nickell) $\Delta x = \Delta y = 0{\cdot}1$	Finite element (Space-time co-ordinates) $\Delta x = \Delta y = 0{\cdot}1$

Table XII. Temperature at $t = 0{\cdot}2$ hr obtained using the finite element technique with $\Delta t = 0{\cdot}1$ hr

1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
1·000	0·944	0·889	0·845	0·845	0·888	0·943	1·000
1·000	0·892	0·776	0·674	0·673	0·774	0·891	1·000
1·000	0·853	0·683	0·519	0·516	0·679	0·852	1·000
1·000	0·853	0·682	0·513	0·510	0·677	0·852	1·000
1·000	0·896	0·784	0·677	0·675	0·780	0·896	1·000
1·000	0·950	0·904	0·859	0·858	0·902	0·950	1·000
1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000

Table XIII. Temperature at $t = 0{\cdot}4$ hr obtained using the finite element technique with $\Delta t = 0{\cdot}1$ hr

1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
1·000	0·988	0·979	0·973	0·973	0·979	0·988	1·000
1·000	0·980	0·964	0·954	0·954	0·964	0·980	1·000
1·000	0·976	0·957	0·943	0·943	0·956	0·976	1·000
1·000	0·976	0·958	0·944	0·944	0·958	0·976	1·000
1·000	0·982	0·968	0·957	0·957	0·968	0·982	1·000
1·000	0·991	0·984	0·978	0·978	0·984	0·991	1·000
1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000

CONCLUSIONS

A solution has been given for two-dimensional linear and non-linear transient heat conduction problems using a finite element weighted residual process. The solution domain was divided into finite elements having space-time co-ordinates. The results for several examples compare favourably with the corresponding analytical and numerical results. This method is flexible in that it does not require constant parameters over the entire solution domain and isoparametric elements[10] can be used. Furthermore, the method is easily programmed, stable, computationally fast, and converges to the exact solution.

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APPENDIX

Notation

A_n = Function identifier
B_n = Function identifier
C_n = Function identifier
c = Specific heat
i = Node number
j = Node number

k = Node number
k_x = Thermal conductivity in x-direction
k_y = Thermal conductivity in y-direction
L_x = Length of solution region in x-direction
L_y = Length of solution region in y-direction
l_x = Direction cosine
l_y = Direction cosine
l = Node number
m = Node number
N_I = Shape function
$N_i, N_j, N_k, N_l, N_m, N_n, N_o, N_p$ = Shape functions
n = Node number
o = Node number
p = Node number
Q = Externally applied heat flux
q = Intensity of heat input
R = Solution domain
R' = Finite element solution domain
S = Boundary surface
S_1, S_2, S_3 = Lengths of boundary surface
S' = External surface area of the finite element solution region
S'_1, S'_2, S'_3 = Segments of external surface area
T = Temperature
T_a = Temperature of the surroundings
$T_i, T_j, T_k, T_l, T_m, T_n, T_o, T_p$ = Value of temperature at appropriate nodal points
T_S = Given boundary surface temperature
t = Time axis
t_c = Distance to centroid in t-direction
x = Co-ordinate axis
x_c = Distance to centroid in x-direction
y = Co-ordinate axis
y_c = Distance to centroid in y-direction
α = Heat transfer coefficient
ζ = t-direction shift of axis to centroid of the rectangular prism element
ξ = x-direction shift of axis to centroid of the rectangular prism element
η = y-direction shift of axis to centroid of the rectangular prism element
ρ = Density
ψ_1, ψ_2 = Constants

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